Cesar

Material models

Ref: CESAR-MU(MM)-v31-EN

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1. Introduction

1.1. Objectives of this document

The choice of a constitutive model adapted to a given structure and to the loads it is subjected to has a decisive influence on the representativeness of the calculation and the results it produces.

During the years 1970-1980, considerable work has been devoted to the development of new constitutive models for geomaterials, and many models were proposed in the literature, that cannot be easily presented in a simple and exhaustive way. However, some of these models offer deformation mechanisms which it may be interesting to bring into play in a particular situation; it is therefore useful to understand which aspects of the actual behavior of materials a given model may or may not represent.

Another important aspect of choosing a model lies in the number of parameters it involves, and in the means of determining their value from the data commonly available in the context of a real project.

The purpose of this document is to provide a presentation of the constitutive models available in the static mechanics modules of CESAR, MCNL and TCNL. It also mentions the models that can be used:

- for concrete at a young age with the MEXO module,
- for dynamic calculations (DYNI),
- for coupled calculations (CSNL and MPNL).

The main goal is to recall the exact mathematical formulation of the models available in the CESAR solver for two-dimensional and three-dimensional mechanical problems, and the list of parameters to provide to use these models.

We hope that this document can also guide the user in choosing a model or another, depending on the problem he has to deal with.

This document partially reproduces the content of previous documents, in particular (but not only):

- technical guide GT n ° 52 from the Studies and Research series of the Laboratories of Bridges and Roads (Mestat, 1993),

- the theoretical summary of version 5 of the software package CESAR-LCPC,

- the documentation of the “user-defined” constitutive models in CESAR.

1.2. Notations used

The formulation of many constitutive models calls for the definition of a certain number of quantities, like the invariants of the stress tensor for example. We define in this section the most useful notations for the continuation. We adopt here the sign convention of the mechanics of continuous media, i.e. stresses are counted positively in traction. Here we summarize the most common notations.

Stresses

We denote by σ_1, σ_2, σ_3 the principal stresses (i.e. the eigenvalues of the stress tensor in the three dimensional space), in the following order :

$$\sigma_1 \geq \sigma_2 \geq \sigma_3.$$

For some models very commonly used in soil and rock mechanics, it will be useful to introduce also the notations Σ_1 and Σ_3 for the largest and the smallest compressive stress in absolute value. For a three dimensional stress state in which all stresses are compressive, the following holds :

$$\Sigma_1 = -\sigma_3 > \Sigma_3 = -\sigma_1 > 0.$$

The mean stress in defined by :

$$p = -\frac{tr(\sigma)}{3}$$

The deviatoric part of the stress tensor σ is the tensor s given by :

$s = \sigma + p\,1$ (or in components : $s_{ij} = \sigma_{ij} + p\,\delta_{ij}$)

Various (scalar) invariants of the stress tensor can be introduced, for instance I_1, I_2, I_3, J_2 and J_3 defined by:

$$I_1 = tr(\sigma) = -3p$$

$$I_2 = J_2 = \frac{1}{2} s_{ij} s_{ij} = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3$$

$$I_3 = \det(s_{ij}) = \sigma_1 \sigma_2 \sigma_3$$

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}$$

In soil mechanics, it is very frequent to use (instead of the second invariant J_2) the (scalar) deviatoric stress q defined by :

$$q = \sqrt{\frac{3}{2} s_{ij} s_{ij}} = \sqrt{3 J_2}$$

For plasticity criteria that depend on the third invariant, one often uses (instead of I_3 or J_3) the so-called Lode angle θ defined by:

$$\theta = \frac{1}{3} \arccos\left(\frac{3\sqrt{3}\, J_3}{2\, J_2^{3/2}}\right)$$

Strains

Some models introduce the deviatoric part of the strain tensor, defined by:

$$\varepsilon_d = \varepsilon - \frac{1}{3} tr(\varepsilon)\, 1$$

Also some models involve a scalar deviatoric strain, defined in the context of triaxial testing ($\Sigma_2=\Sigma_3$ and $\varepsilon_2=\varepsilon_3$) by :

$$\varepsilon_d = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$$

and in a general three-dimensional context by :

$$\varepsilon_d = \sqrt{\frac{2}{3} (\varepsilon_d)_{ij} (\varepsilon_d)_{ij}}$$

Moreover, the volumetric strain, denoted by ε_v, is equal to the trace of the strain tensor ε.

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Total stresses and effective stresses

Most models in soil mechanics rely on the effective stress principle proposed by Terzaghi, which states that the strains are controlled by the variations of the effective stresses rather than the total stresses. With the convention adopted here for the stresses, the effective stress tensor σ' is defined by :

$$\sigma'_{ij} = \sigma_{ij} + u\delta_{ij}$$

where u denotes the pore pressure.

In the computation module MCNL, in most cases, it is implicitly assumed that the computations is carried out with the effective stresses in drained conditions. It is also possible to make a computation in total stresses, or to perform a simulation in undrained conditions, by means of a specific procedure described hereafter in the framework of the so called « user defined » models.

NB. In some cases, in rock mechanics for instance, effective stresses can be defined by :

$\sigma'_{ij} = \sigma_{ij} + b\, u\, \delta_{ij}$

where the scalar coefficient b can be different from 1 if the variation of the soil grains is not negligible. This cannot be taken into account in CESAR except in the MPNL computation module.

2. Phenomena and models

2.1. Elastic behaviour and elastoplastic behaviour

A constitutive model is a set of equations which represent the relation between the stresses undergone by the material, generally denoted by σ, and the deformations which result from it, often denoted by ε within the framework of the small strains and small displacements.

The constitutive model is a local relation, at the level of the "material point" of the macroscopic modelling, between stresses and strains. Depending on the scale adopted for the modelling, the behaviour of the same material will therefore not necessarily be represented by the same model (s).

On the other hand, the constitutive model is a mathematical representation which must give the best account of the physical phenomena responsible for the deformation (dislocations in a crystalline medium, reduction of porosity in geomaterials, loss of stiffness in concrete due to cracking, etc.). The same phenomenon, in different contexts, can be represented by different models. Some phenomena can hardly or not at all be taken into account with an inappropriate model. In the following, an attempt is made to relate families of models with types of phenomena.

In a very general way, the behaviour of a material can depend on a multitude of factors, time, the speed at which it is loaded, the history of the strains and stresses it has undergone. An entirely general formulation is therefore unusable in practice, and one is thus led to simplify the real behaviour to get a formulation which can be used for the practical applications. The constitutive model is therefore only a simplified mathematical representation of reality, which can be more or less faithful; in this sense, it is better to speak of a constitutive model rather than a constitutive “law”.

For the three-dimensional continuous medium, the simplest behaviour is the linear elastic behaviour. In this case, stresses are directly proportional to strains:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

where C_{ijkl} are the components of the tensor of the elastic moduli C, assumed to be constant in the case of linear elasticity. Taking into account the symmetries of the tensor of the elasticity modules, the linear elasticity is characterized by 21 independent coefficients.

It is very often assumed, although it is only rarely justifiable in soil mechanics, that the behaviour of the material is isotropic, i.e. the deformability properties of the material are identical in all directions. In this case, for linear elasticity, the elasticity tensor C is only characterized by two independent scalar parameters, the Young’s modulus and the Poisson's ratio.

Regardless of its isotropic nature or not, the linear elastic behaviour is reversible: if stresses are reduced to zero, the strain vanishes. On the other hand, there is no limit to the value of the stresses that can be applied to the material: its strength is therefore infinite (it will be noted that if the stresses become arbitrarily large, we will end up leaving the domain of small strains, and it would therefore be necessary to modify the constitutive model).

In addition, the elastic behaviour is not necessarily linear. In the nonlinear elastic case, we will write

$$d\sigma_{ij} = C_{ijkl}\,d\varepsilon_{lk}^{e}$$

where the C_{ijkl} coefficients are the tangent elastic moduli, which are no longer constant but depend on strains or stresses (depending on the model).

The behaviour of some materials goes beyond the reversible domain: in geomaterials, in particular, it is clear that strains do not always disappear when the material is unloaded. On the other hand, practical experience indicates that the load that can be applied to a structure is not infinite, and simple experiments show that there is therefore a limit to the stresses that the material can support. The elastoplastic framework provides a way to describe these two realities. One distinguishes, in the deformation of the material, an “elastic” (reversible) part, noted ε^e, which vanishes if one removes the stresses applied to the material, and a “plastic” part, noted ε^p, which is irreversible or permanent, in the sense that it subsists once the applied constraints have disappeared:

$\varepsilon = \varepsilon^e + \varepsilon^p$

In the framework of elastoplasticity, elastic strains are related to the stresses by

$\sigma_{ij} = C_{ijkl}\varepsilon^e_{kl} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})$

And it remains to stipulate:

- Under what circumstances the plastic part of the strain tensors evolves,
- And how its evolutions can be related to the applied loads.

The situations in which the plastic strain can evolve are described by means of a scalar function of the stress tensor $f(\sigma)$, called plasticity criterion or yield function, such that :

- if $f(\sigma) < 0$: strains are reversible and the plastic part does not evolve
- if $f(\sigma) = 0$: the plastic part can evolve.

The condition $f(\sigma) < 0$ defines a domain in the stress space such that : if the stress state remains inside the domain, strain remain reversible. The stress state cannot move outside this domain, but if it is on the boundary of the domaine, plastic strain can occur. The boundary of the domain defined by $f \leq 0$ is often called "yield surface", The mathematical expression of the function used a plasticity criterion and the parameters is involves depend on the material and must be identified on the basis of experimental tests and of mathematical assumptions : for instance, if the material is assumed to behave isotropically, the value of the function f only depends on the invariants of the stress tensor.

If the stress state reaches the yield surface, plastic strains can evolve. Their evolution is described by the following relation, called "plastic flow rule":

$d\varepsilon^p = d\lambda\ \partial g/\partial\sigma$

where $g(\sigma)$ is another scalar function of the stress tensor, called "plastic potential". Its derivative with respect to σ defines the direction of the plastic strain increment, their magnitude depending on the scalar parameter $d\lambda$, called "plastic multiplier", which is bound to be non-negative.

If the plastic potential is the same function as the yield function, the flow rule is said to be "associated". In geotechnics, this type of model may not describe satisfactorily the actual behaviour. The plastic strains are defined by a function g different from the plastic criterion, and has to be identified on the basis of experiments.

From a mathematical point of view, one gets a closed problem by adding a condition called "consistency condition", which expresses the fact that the stress state cannot lie outside the yield surface. If the increment of plastic strain is non zero, the value of the yield function must remain equal to zero, which can be expressed by:

$d\lambda\ df = 0$

Elastoplasticity makes it possible to account for the appearance of irreversible deformations and the fact that the local resistance of the materials is not infinite. However, this behaviour remains relatively simple: in particular, it does not explicitly involve time. The speed at which mechanical loading is applied has no influence on the strains. For some materials (polymers, bitumen) this assumption is not realistic. On the other hand, in the formalism of elastoplasticity, the representative point of the stress state σ_{ij} cannot go outside the elastic domain. In the formalism of viscoplasticity, on the other hand, the state of stresses can leave the elastic field, but the MCNL module does not take into account this type of behavior.

Note: The expressions above assume that the yield function and the plastic potential are sufficiently regular so that one can calculate their derivatives: this is not always the case, even for common criteria (like the Mohr-Coulomb model), and the numerical treatment may require specific precautions.

Perfect plasticity and plasticity with hardening

Numerous experimental observations show that the elastic domain can evolve during the transformations undergone by a material. One is thus led to introduce into the expression of the criterion one or more "hardening" parameters which control the size and the shape of the elastic field, and the model must provide a suitable description of their evolution.

The physical interpretation of the phenomenon of hardening falls outside the scope of this document, but it can be pointed out that various theories were proposed to distinguish in the not restorable part of the mechanical power provided to the material a fraction dissipated by heat and a fraction involved in the transformation of the material associated with hardening.

In practice, the plasticity criterion is described by a function $f(\sigma, \zeta)$, where ζ is not a constant but may vary, in a way that should then be specified. One way of obtaining a mathematically well-posed problem is to describe the variations of ζ by relating them to a parameter characteristic of the evolution of the material. It can be for example of a law connecting the parameter of hardening to the plastic strain, such as:

$$\zeta = F(\varepsilon^p) \quad \text{or} \quad \dot{\zeta} = A(\sigma_{ij}, \varepsilon^p_{ij})\, \dot{\varepsilon}^p_{ij}$$

The family of models called "Cam-Clay" can be formulated in this context. The hardening law can also relate the rate of the hardening parameter(s) $\dot{\zeta}$ to the plastic dissipation :

$$\dot{\zeta} = B(\sigma_{ij}\dot{\varepsilon}^p_{ij})$$

The plastic multiplier is determined using the consistency condition, which now takes into account the possible variations of ζ:

$$\dot{\lambda} \geq 0 \text{ and } \dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial \zeta}\dot{\zeta} = 0 \text{ if } \dot{\lambda} > 0$$

Multi-mechanism models

More complex models than those described by the formalism above can be found in the literature, in which the elastic domain is defined by two criteria f1 and f2 (or more): the load point is inside the elastic domain if $f1(\sigma) < 0$ and $f2(\sigma) < 0$.

Each of the yield surfaces associated with the two criteria can be fixed or associated with hardening. The calculation of plastic strains can involve, for these models, two independent plastic multipliers, and the plastic strain increment is a linear combination of the normals of the two surfaces. Plastic multipliers are determined by writing a consistency condition for each of the criteria.

2.2. Shear strength and compression strength

The presentation given above is fairly mathematical, but it is based on experimental observations: there is a threshold, in the stress space, beyond which strains are no longer reversible.

The first practical step to formulate an elastoplastic model consists in identifying this threshold, that is to say the states of stresses likely to bring about the failure of the material.

Experimentally, this threshold can be defined in certain situations. At the end of the 19th and the beginning of the 20th century, Tresca and von Mises observed that materials do not support the stress states in which certain material facets are subjected to high shear stresses, or, what amounts to the same thing, the stress states in which the principal stresses are very different. In other words, the materials are sensitive to the deviatoric component of the stress tensor, or to the maximum shear stress.

For the two criteria proposed by Tresca and von Mises, the value of the yield function does not depend on the value of the mean stress.

This feature fairly well represents the strength of metals or clays. But these models are not well suited to describe the strength of materials like sands, which can withstand greater deviatoric stress when the mean stress increases. To take this observation into account, more complex criteria can be adopted, such as the

Mohr-Coulomb or Drucker-Prager criteria, which make it possible to account for the observation that the maximum shear stress, during a triaxial compression test by example, is larger if the confining pressure increases.

Just as one can demonstrate experimentally the existence of a maximum value of the shear stress that a material facet can support, one can also show that certain materials exhibit an irreversible behavior when they are subjected to excessive compression: for example, sediments undergo ab irreversible reduction in porosity during the formation of sedimentary basins. This influence of the mean stress can be demonstrated in an oedometer or during an isotropic compression test.

It is understood that the models (that is to say, in the elastoplastic framework, the plastic criteria and plastic potentials) which represent these mechanical properties are different. For an isotropic material, in the case of the shear strength, it is the distance between the stress state and the axis of the isotropic compression axis in the principal stresses space that governs the onset of plastic strains. In the case of compression strength, it is rather the projection of the stress state on the isotropic compression axis that can be used as a basis for the construction of a plastic criterion. In any event, the mathematical formulation of the yield function aims at accounting for a phenomenon that can be characterized experimentally.

2.3. Hardening

For a perfectly plastic material, the stress tensor cannot lie outside the initial yield surface, because the elastic domain does not evolve.

With hardening models, the surface is likely to evolve. A family of simple models consists in taking into account an "isotropic" work hardening in the sense that the actual yield surface is deduced from the initial by an homothetic transformation in the stress space. In general, this evolution corresponds to an increase in the size of the elastic field (positive hardening).

In soil mechanics, this evolution is most often limited and the yield surface evolves to a limit surface, called failure surface.

Other models rather propose a "kinematic" type of hardening: the yield surface is translated without being deformed in the stress space. These models make it possible to account for phenomena observed in metals (Bauschinger effect).

On the other hand, in some models, the size of the elastic domain decreases during plastic deformation. We talk about negative hardening (or softening). This type of model poses theoretical problems, because the solution to the problem ceases to be unique. From a numerical point of view, one generally observes a strong dependency of the solution obtained by finite elements with the size of the elements of the mesh; the solution shows “shear bands” where strong deformations are localized, and one can discuss the representativeness of the solution found.

2.4. Undrained shear strength

An important aspect of soil behavior is the influence of water in the soil deformability. A number of laboratory tests (in particular on the triaxial apparatus) are carried out in undrained condition. Provided that the pressure of the water in the sample can be measured during the test, this type of experiment makes it possible to obtain information useful for the characterization of the initial yield surface.

Modeling the behavior of soils in undrained conditions poses specific difficulties: in particular, certain constitutive models lead to infinite undrained shear strength. This is what motivated the development of critical state models (like the Cam-Clay model). The use of undrained calculations must therefore be conducted by a user warned of these theoretical difficulties.

2.5. Small-strain behaviour

A large part of the ground surrounding a geotechnical structure does not reach failure, nor the criterion of plasticity. The global response of the ground mass is therefore not always well represented if the model that is used to represent the behavior of areas subjected to small deformations is not suitable. Elastic nonlinear models have been proposed to take into account precisely the response of soils under small strains.

2.6. Anisotropy

In general, an isotropic material is a material whose response does not depend on the orientation of the stresses which it undergoes. The anisotropy of a material can be the consequence of its structure (schistosity, composite made up of a matrix and fibers oriented in a given direction) and / or the history of the stresses which it underwent.

Taking full account of the anisotropy of material properties poses difficult theoretical problems (in particular, the notion of isotropy relates to a given geometrical configuration).

We just recall here the most common approaches for taking into account the anisotropy in the elastic domain and in the plastic domain.

In the cases, if the behaviour of the material is not the same one in all the directions of space, it is necessary to define a set of "material directions", which make it possible to take into account the orientation of the loading with respect to the material.

2.6.1. Anisotropy of elastic properties

In the linear isotropic case, the stress tensor can be computed for a given strain tensor by a relation involving two material parameters, the Young's modulus and the Poisson's ratio (or the two Lamé coefficients, or the compression modulus K and the shear modulus G, according to the formulations). The two tensors have the same main directions.

The simplest anisotropic case corresponds to the case of transverse isotropy. This situation can account for the effect of the consolidation of sediments, in which the vertical direction plays a special role, which can result in a different stiffness in the vertical and horizontal directions.

The behavior is then characterized by 5 independent coefficients.

The other anisotropic behaviour that can be encountered is the orthotropic behaviour. The material presents material symmetries attached to three orthogonal planes. The behaviour is characterized by nine independent coefficients.

This type of symmetry rather corresponds to artificial materials, such as composite materials, for which one can try to derive the model parameters from theoretical considerations of homogenization (and / or auxiliary numerical calculations).

In some models, the directions of anisotropy of the elastic properties depend on the history of loading: CESAR only proposes anisotropic models in which these directions, for a given point, are fixed (they can however in certain special cases vary from point to point).

2.6.2. Anisotropy of plastic properties

Structural anisotropy – Induced anisotropy

In a masonry wall, the brickwork defines specific directions and results in an anisotropic macroscopic behavior. The directions of anisotropy reflect the structure of the material, and do not change over time.

In a material such as rolled steel, the anisotropy of the strength properties results from the process for obtaining the steel plate. The directions of anisotropy are fixed and generally do not evolve under normal conditions of use of steel.

In these two cases, the orientation of the material directions is defined once and for all (at a given point): this case is referred to as structural anisotropy.

One can imagine models in which the loading applied to the material induces an anisotropy (compared to the current configuration): the orientation of the material directions varies under the effect of the applied loads. A family of models which produces this kind of effects is obtained by using a kinematic hardening. For clays, different authors have proposed specific, more complex, formulations of the plastic properties. This case is referred to as induced anisotropy (by loading). This type of model is not very widespread and difficult to use, the identification of the parameters being generally difficult.

2.7. Damage

The stiffness of some materials (their Young's modulus in the isotropic case) decreases when the deformation they undergo exceeds a certain threshold. This is the case, for example, of concrete, in which cracking develops during deformation. On the other hand, this type of material cannot bear a stress beyond a given threshold. Although the models are different, as in the case of negative hardening, such models involve a loss of the uniqueness of the solution and a strong dependence of the solution obtained by finite elements to the size of the elements of the mesh.

To describe the behaviour of such materials, there are a wide variety of damage models. The approach is somewhat to that of elastoplasticity: a criterion determines when the elastic modulus begins to decrease: it is generally a threshold on the deformations undergone by the material; then a specific model describes this decrease.

In the simplest case of the isotropic damage, the tensor of elastic moduli of the damaged material is equal to its initial value multiplied by a scalar coefficient lower than 1, noted 1-d, where d is the damage variable. The model must specify how this variable evolves when the damage threshold is reached.

2.8. Other phenomena

This section mentions a few other phenomena that are generally not completely or not directly handled by CESAR.

2.8.1. Creep and relaxation, time-dependent behaviour

Creep is a delayed behaviour of certain materials, in which the deformations can evolve under a constant stress. This type of behaviour goes beyond the framework of elasticity or elastoplasticity, which are "instantaneous" behaviours.

Relaxation corresponds to the situation in which the stresses in a solid subjected to a given deformation decrease over time (which can be responsible for a pre-stress in concrete).

Depending on whether the deferred deformations are reversible or not, these two types of behavior can be represented by quite different models (viscoelastic or viscoplastic).

This type of behaviour is practically not taken into account in CESAR, even if one can, in certain cases, represent a reduction in the elastic modulus of a material (for instance by means of the loading option EFD or of specific material models for beams for instance).

2.8.2. Aging, corrosion, chemical or thermal deformation

So far, we have not discussed situations in which the behaviour of a material varies over time, because the material ages. This type of variation can be the result of chemical effects, for example. We can in particular think of concrete, the module of which changes over time after setting. We can also mention the case of corrosion of metallic materials. These very specific aspects are not covered in this manual.

For concrete, precise modeling of the behavior at a young age of the material, and internal reactions likely to lead to its degradation, are available in dedicated modules (TEXO, MEXO, RGIB).

We can also mention that a specific study of the corrosion of steel strips in mechanically stabilized earth walls has been conducted with CESAR (Chau, 2010).

2.8.3. Large strains

This document does not cover the case of large strain that can occur in soft soils (steel and concrete structures are in principle designed to remain far from this domain).

3. Bulk elements in statics

3.1. Classical models and user-defined models

In the computation module MCNL of CESAR, originally each constitutive model was the combination of an elastic law (generally linear and homogeneous isotropic), a criterion of plasticity, a plastic potential, and in some cases of a hardening law. In this approach, described in the following as "classical", these elements define a combination associated with a given value of the IMOD indicator (see the reference manual of the solver, for the ELEM module). For example, the "Mohr-Coulomb model", with an isotropic linear elasticity and without hardening, corresponds to the value IMOD = 10. It represents in a synthetic way, by only one indicator, all aspects of the constitutive model (and even properties that are not strictly related to the material deformability, such as its density).

In a second step, another class of constitutive models has been developed to give the user more possibilities:

- choosing an elastic model (linear or not, isotropic or not, with homogeneous parameters or not) independently of the plasticity criterion,

- being able to compare the differences between several hardening models for the same criterion,

- combining two plastic mechanisms,

- making calculations in drained or undrained condition, etc.

The multiplicity of possible combinations then made it difficult to associate a single indicator with each of these combinations, which led to a second implementation of the constitutive models in CESAR, called "user-defined models". In this context, the constitutive model is seen as the combination of several distinct sub-models referred to by specific sub-indicators.

The same mathematical formulation can be implemented in either context. However, for convenience, the rest of the text is organized according to the two classes of models.

3.2. Classical elastic and elastoplastic models

In the vast majority of constitutive models of this class in CESAR-LCPC, the elastic part of the model is linear and isotropic with moduli independent of the point considered, so that the tensor C is constant and characterized by two scalars, the Young's modulus E and the Poisson's ratio ν. There are a few exception: for example the transverse isotropic elastic model IMOD = 2 and the elasticity model with isotropic dilatancy IMOD = 88.

3.2.1. Linear isotropic elasticity (IMOD=1)

The case of linear elasticity corresponds to the situation in which the variations of the strain tensor are proportional to those of the stress tensor. This model is still very often used to analyze the behaviour of soil masses and structures. The behaviour is isotropic if, in addition, all the directions of space are equivalent.

Mathematically, two formulations are commonly used to represent the isotropic linear elastic behaviour:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\,\mu\varepsilon_{ij}$$

Or, intrinsic notations:

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\operatorname{tr}(\sigma)\;1 \qquad \sigma = \lambda\;\operatorname{tr}(\varepsilon)\,1 + 2\,\mu\varepsilon$$

where E denotes Young's modulus, ν Poisson's ratio, λ and μ the Lamé coefficients. E, λ and μ have the dimension of a force by unit surface (there are expressed in N/m^2 or in Pa), while ν is a dimensionless quantity. The following relations connect (E,ν) and (λ,μ) :

$$\lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

It is also possible to use coefficients G and K and to formulate the model as follows:

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + 2G\varepsilon^{d}_{ij}$$

where ε^d is the deviatoric part of the strain tensor. On can show that:

$$s_{ij} = 2G\varepsilon^{d}_{ij} \qquad p = -3K\;\varepsilon_{kk}$$

The shear modulus G coincides with μ, and the compression modulus K is such that : $3K=3\lambda+2\mu$.

For linear isotropic elasticity, in CESAR, the user inputs the values of E and ν.

Parameters for the linear isotropic elastic model (IMOD=1)

The linear isotropic elastic model, as well as the elastic part of most classic constitutive models in CESAR is defined by the input of the following parameters:

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]

3.2.2. Linear transversely isotropic elasticity (IMOD=2)

In the isotropic case, applying two stress increments deduced from each other by a rotation leads to strain increments deduced from each other by the same rotation: in other words, the material has the same stiffness characteristics in all directions of space. In particular, the stress and strain increments always have the same principal directions and the same eigenvectors.

The CESAR solver proposes a transversely isotropic linear elastic model, described in the theoretical Reference of families 01 and 02: in this model, the material exhibits a revolution symmetry axis.

Model formulation

We denote by e_v a unit vector of the axis of symmetry, and (e_{h1}, e_{h2}) two vectors of the plane perpendicular to e_v such that (e_{h1} , e_{h2} , e_v) forms a direct orthonormal frame.

By noting σ_{ij} and ε_{ij} the components of the tensors σ and ε in (e_{h1} , e_{h2} , e_v), the transverse isotropic behavior is formulated as follows:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} 1/E_h & -\nu_h/E_h & -\nu_v/E_v & 0 & 0 & 0 \\ -\nu_h/E_h & 1/E_h & -\nu_v/E_v & 0 & 0 & 0 \\ -\nu_v/E_v & -\nu_v/E_v & 1/E_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu_h)/E_h \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

with :

E_h Young's modulus in the isotropic plane (or transverse Young's modulus)

E_v Young's modulus in the direction of the symmetry axis (or longitudinal Young's modulus)

ν_h Poisson's ratio in the isotropic plane

ν_v transverse Poisson's ratio

G shear modulus in a plane containing the symmetry axis (between the directions of e_v and of a perpendicular vector)

The model is defined by 5 parameters E_h, E_v, ν_h, ν_v and G, and the unit vector e_v of the symmetry axis.

En bidimensional condition and in plane strain, the direction e_v of the symmetry axis is in the plane of the mesh. It is assumed that the vector e_{h2} is also in this plane, so that e_{h1} is normal to the mesh plane. The orientation of (e_{h1} e_{h2} e_v) is defined by the angle between Ox and e_{h2} :

$$e_{h1} = e_z$$

$$e_{h2} = \cos\theta\, e_x + \sin\theta\, e_y$$

$$e_v = -\sin\theta\, e_x + \cos\theta\, e_y$$

where (e_x, e_y) is the basis of the Cartesian frame used for the nodes coordinates and e_z is normal to the mesh plane.

In three-dimensional condition, the direction e_v of the symmetry axis is defined by two Euler angles θ and φ defined in the figure below.

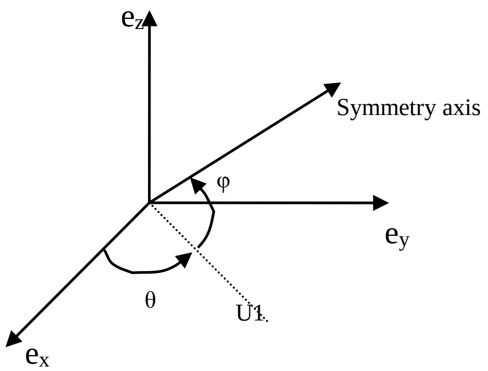


Figure 1 – Definition of the symmetry axis

Or precisely:

$u_1 = \cos\theta e_x + \sin\theta e_y$ $\qquad u_2 = -\sin\theta\, e_x + \cos\theta\, e_y$ $\qquad u_3 = e_z$

$e_v = \cos\varphi\, u_1 + \sin\varphi\, u_3$ $\qquad e_{h1} = u_2$ $\qquad e_{h2} = e_v \wedge e_{h1} = -\sin\varphi\, u_1 + \cos\varphi\, u_3$

which leads to:

$e_{h1} = -\sin\theta\, e_x + \cos\theta\, e_y\,;$

$e_{h2} = -\sin\varphi \quad \cos\theta\, e_x - \sin\varphi \quad \sin\theta\, e_y + \cos\varphi\, e_z$

$e_v = \cos\varphi \quad \cos\theta\, e_x + \cos\varphi \quad \sin\theta\, e_y + \sin\varphi\, e_z\,;$

Parameters for the transversely isotropic linear elastic model (IMOD=2)

- density (RO) [kg m^{-3}]
- Young's modulus in the isotropic plane E_h (E1) [Pa]
- Young's modulus in the direction of the symmetric axis E_v (E2) [Pa]
- Poisson's ratios in the isotropic plane ν_h (P1) [-] and transverse ν_v (P2) [-]
- shear modulus G (G2) [Pa]
- in 2D:

    angle θ between Ox and the isotropic plane (TETA) [deg]
- in 3D :

    θ and φ angles defining the direction of the symmetry axis (TETA, PHI) [deg]

3.2.3. Linear orthotropic elasticity

For the applications that motivated the development of CESAR, it was not found necessary to implement an orthotropic elasticity. This model is available in the framework of the "user-defined models" (see 3.3.3.7).

3.2.4. "Isotropic-dilatancy" elastic model (IMOD=88)

In this model was proposed by R. Frank (1974), an elastic volumetric strain is caused by a deviatoric stress, in addition to the volume change due to the mean stress :

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} - k\,\Delta\tau$$

where : $\Delta\tau = \sqrt{\frac{2}{3}J_2}$

This formulation is actually non-linear. In the implementation, the model was modified into :

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} - k_A\Delta\tau - k_B\,I_1 - k_C\,I_3$$

Making it possible to introduce corrections associated with the stress invariants I_1 and I_3.

Parameters for the "Isotropic-dilatancy" elastic model (IMOD=88)

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- coefficient for shear k_A (XKA) [Pa^{-1}]
- coefficient for mean stress k_B (XKB) [Pa^{-1}]
- coefficient for third stress invariant k_c (XKC) [Pa^{-1}]

3.2.5. Early-age concrete behaviour (IMOD=5)

Modelling early age concrete behaviour accounts for phenomena induced by the chemical reactions that take place during setting: exothermal reactions and maturation of the material. The constitutive model links the thermal and mechanical behaviours during time and the evolution of the elastic moduli as the reactions develop. In practice in CESAR, there is a sequence of a specific diffusion computation using module TEXO and a mechanical computation using module MEXO.

The hydration reaction in concrete results, at the macroscopic level, in an increase in the quantity of hydrates, an evolution of the stiffness, an endogeneous shrinkage due to dessication (consequence of the Le Chatelier contraction) and thermally induced strains. Considering that the material is elastic and neglecting the effects of creep, the stress increment is given by:

$$d\sigma = 2\,G(\xi)\,de \;+\; K(\xi)\,[\,d\varepsilon - 3\,\alpha\,dT - 3\,\beta\,d\xi]\;\;1$$

where ε denotes the trace of the strain tensor and e its deviatoric part.

$K(\xi)$ denote $G(\xi)$ respectively the bulk modulus (or compression modulus) and the shear modulus, α is the linear thermal dilatation coefficient and β the coefficient of chemical dilatation / shrinkage. In first approximation, α and β can be considered as constants. Coefficient β is there for the total endogeneous shrinkage measured for a given formulation.

The bulk and shear moduli depend on the hydration degree through the Young's modulus through:

$$K(\xi) = \frac{E(\xi)}{3(1-2\nu)} \quad \text{and} \quad G(\xi) = \frac{E(\xi)}{2(1+\nu)}$$

where Poisson's ratio is assumed to be constant. The evolution of Poisson's ratio is given by the following relation (adapted from Byfors' law):

$$E(\xi) = E_\infty \;\; f(\xi) \;\; \text{with} \;\; f(\xi) = \frac{1+1{,}37 R_{c\infty}^{\;2,204}}{1+1{,}37 R_c(\xi)^{2,204}} \left[\frac{R_{c\infty}}{R_c(\xi)}\right]^{2,675} \;\; \text{and} \;\; R_{c\infty} = \left[\frac{E_\infty}{7250}\right]^{\frac{1}{0,471}}$$

where E_∞ and $R_{c\infty}$ represent respectively le Young's modulus and the compression strength of the hardened material. The compression strength $R_{c\infty}(\xi)$ is described by a bilinear function of ξ (ξ_0 is a threshold value of the hydration degree beyond which the material is considered as solid):

$$R_c(\xi) = \begin{cases} \xi R_{co} \quad \text{si} \quad \xi \le \xi_0 \quad \text{avec} \quad R_{co} = \xi_0 R_{c\infty}/10 \\ (R_{c\infty} - R_{co})\dfrac{\xi-\xi_0}{1-\xi_0} + R_{co} \quad \text{si} \quad \xi > \xi_0 \end{cases}$$

Parameters for the « early-age concrete » model (IMOD=5)

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- linear thermal dilatation coefficient α (DILAT) [K^{-1}]
- final endogeneous shrinkage of concrete β (RETRA) [-]
- theshold value of the hydration degree $\xi°$ (SEUIL) [-]
- hydration degree (HYD) : this parameter is set to zero for a chemically active material, and to a non zero value for a chemically inactive material : in this case, the hydration degree is equal to a fixed value HYD (the corresponding group must be declared as inactive in the TEXO computation using option INA).

3.2.6. Mohr Coulomb without hardening model (IMOD=10)

The Mohr-Coulomb criterion is the most classically used criterion in soil mechanics, originating from the work of Coulomb on the stability and failure of structures. It expresses the fact that the tangential stress τ on a material facet is limited by a value that depends on the normal stress σ, which is exerted on this facet: $|\tau| \leq c + \sigma \tan \varphi$. Below, an equivalent formulation using principal stresses is provided.

It should be noted that the Mohr-Coulomb criterion allows matching the criterion introduced by H. Tresca in 1864 for metals when the friction angle φ is set to zero.

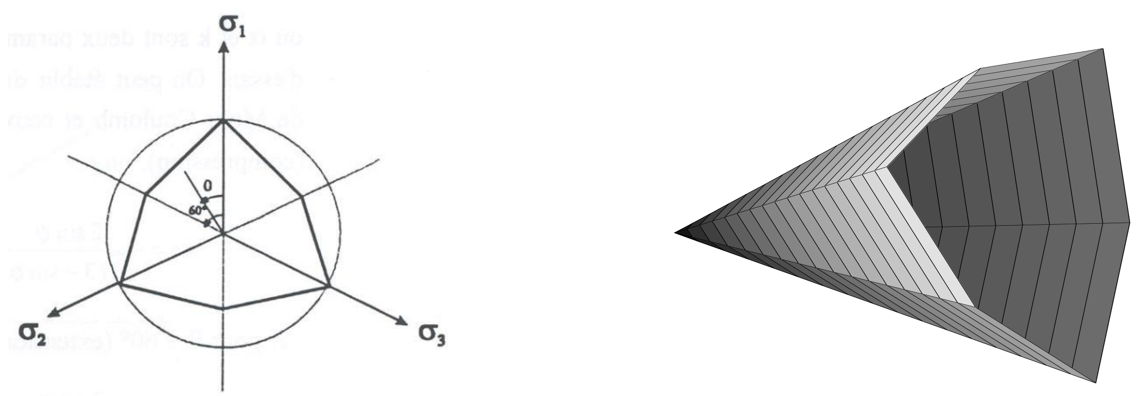
The Mohr-Coulomb model (IMOD=10) of CESAR is an elastic-perfectly plastic model.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

Le critère de plasticité est celui de Mohr-Coulomb, qui s'écrit, dans le contexte de la mécanique des sols :

$$f(\sigma) = \Sigma_1 - \Sigma_3 - (\Sigma_1 + \Sigma_3) \sin \varphi - 2 c \cos \varphi$$

where Σ_1 and Σ_3 represent respectively the largest and the smallest of the principal stresses, positive in compression. It should be noted that the value of the criterion does not depend on the intermediary principal stress Σ_2.



in the deviatoric plane in the principal stresses space

Figure 2 - Visualization of the yield surface for the Mohr-Coulomb model (after Lee, 1994)

The parameter c is termed material cohesion and the angle φ is called the internal friction angle. When $\varphi = 0$, the Mohr-Coulomb criterion matches the Tresca criterion, which specifies that the difference between the two principal stresses cannot exceed a limit value equal to two times the cohesion:

$$f_{TRESCA}(\sigma) = \text{Sup} |\sigma_i - \sigma_j| - 2 c$$

The Mohr-Coulomb criterion is generally used with a plastic potential of the same form, but with an angle different from the friction angle:

$$g(\sigma) = \Sigma_1 - \Sigma_3 + (\Sigma_1 + \Sigma_3) \sin \psi + \text{constante}$$

where ψ is the dilatancy angle ($\psi = \varphi$ if the flow rule is associated).

It should be noted that when the flow rule is associated and the friction angle is non-zero, the flow rule leads to an irreversible increase in the volume controlled by the friction angle and usually not realistic for significant shear strains: this model does not properly describe the fact that, at large strains, the shearing of a soil generally occurs at constant volume (this type of experimental observation has led to the introduction of the concept of limit states in soil mechanics).

Parameters for the Mohr-Coulomb model (IMOD=10)

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- cohesion c (C) [Pa]
- friction angleφ (PHI) [deg]
- dilatancy angle ψ (PSI) [deg]

3.2.7. Von Mises without hardening model (IMOD=11)

The von Mises criterion has been proposed independently by several authors, including von Mises in 1913, following experiments carried out on metals. It is a criterion based on the deviatoric stress.

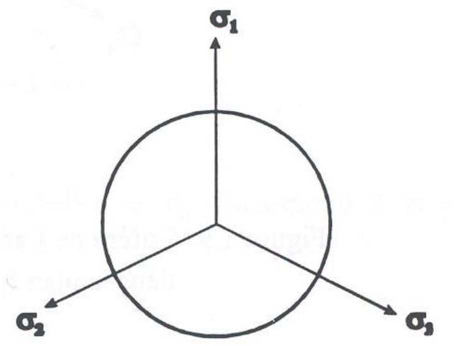
The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

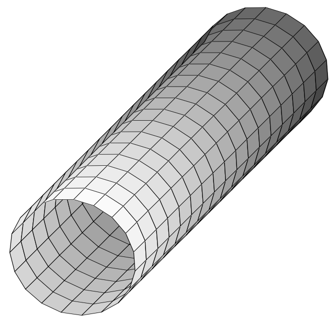
The criterion proposed by von Mises writes simply as a function of the second invariant of the stress tensor J_2 :

$$f(\sigma) = J_2 - k^2 = \frac{1}{2} s_{ij}\, s_{ij} - k^2$$

The criterion depends only depend on one material parameter k, which represents the shear strength. A simple calculation shows that the tensile strength is equal to $k\sqrt{3}$.

The flow rule is associated.





in the deviatoric plane   in the principal stresses space

Figure 3 - Visualization of the yield surface for the Von Mises (after Lee, 1994)

Parameters for the von Mises without hardening model (IMOD=11) :

- density (`RO`) [kg m^{-3}]
- Young's modulus (`YOUNG`) [Pa]
- Poisson's ratio ν (`POISS`) [-]
- shear strength k (`K`) [Pa]

3.2.8. von Mises with hardening model (IMOD=12)

It is an elastoplastic model with hardening.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion is again the one proposed by von Mises (as for the previous model):

$$f(\sigma) = J_2 - k^2 = \frac{1}{2} s_{ij} s_{ij} - k^2$$

but the shear strength k can evolve as plastic strains develop.

The flow rule is associated.

The hardening law relates the shear strength k to the plastic dissipation rate dW^p, defined by :

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^p$$

The evolution of k is described by:

$$k + \Delta k = \sqrt{k^2 + 2H \Delta W^p}$$

where Δk and ΔW^p denote the increments of k and W^p. According to the sign of H, this model allows representing a classical hardening (positive hardening: the elastic domain increases), or a strength loss (negative hardening: the elastic domain shrinks). The determination of the parameter H is based on the study of triaxial tests (Ricard, 1975; Yuritzin, 1981): H corresponds to the constant slope of the (ε^p, σ) curve for a uniaxial test.

Parameters for the von Mises with hardening model (IMOD=12) :

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- initial shear strength k (K) [Pa]
- parameter H of the hardening law (H) [Pa]

3.2.9. Drucker Prager without hardening model (IMOD=13)

The Drucker-Prager criterion (1952) is a generalisation of the von Mises criterion to granular materials, taking into account the first invariant of the stress tensor J1 and the second invariant of the deviatoric stress tensor J2.

It is an elastic-perfectly plastic model.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

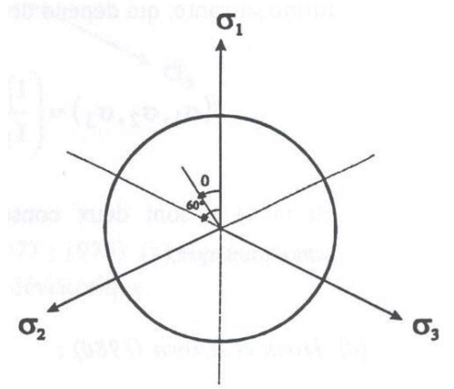
The plasticity criterion is given by :

$$f(\sigma) = \sqrt{J_2} + \alpha\, I_1 - k$$

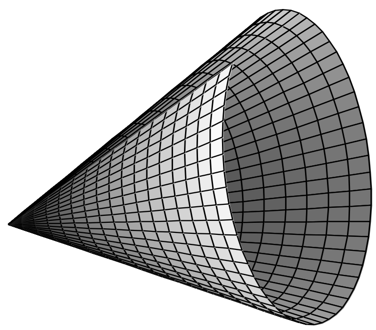
The plastic potential has the same form :

$$g(\sigma) = \sqrt{J_2} + \beta\, I_1 + \text{constante}$$

The flow rule is associated if $\alpha = \beta$. If $\alpha = \beta = 0$, les paramètres α and β sont nuls, the criterion matches the von Mises model.



in the deviatoric plane



in the principal stresses space

Figure 4 - Visualization of the yield surface for the Drucker-Prager (after Lee, 1994)

In practice, CESAR kernel does not use parameters α, β and k, but three other parameters c, φ and ψ linked to the former by:

$$\alpha = \frac{2\sin\varphi}{\sqrt{3}(3-\sin\varphi)} \quad \beta = \frac{2\sin\psi}{\sqrt{3}(3-\sin\psi)} \qquad k = \frac{6c\cos\varphi}{\sqrt{3}(3-\sin\varphi)}$$

This model makes it possible to fit the Mohr-Coulomb model with the Drucker-Prager model for the simulation of triaxial tests of revolution in compression ($0 > \sigma_2=\sigma_3 > \sigma_1$). It should be noted that it does not allow obtaining identical results for the two models for a triaxial test in tension ($0 > \sigma_1 > \sigma_2=\sigma_3$) or a shear test in plane strains ($\varepsilon_2=0$).

Parameter c is called cohesion, angles φ and ψ are the friction angle and the dilatation angle.

Parameters for the Drucker Prager without hardening model (IMOD=13) :

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- cohesion c (C) [Pa]
- friction angle φ (PHI) [deg]
- dilatancy angle ψ (PSI) [deg]

3.2.10. Drucker Prager with hardening model (IMOD=14)

It is an elastoplastic model with hardening.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion is given by :

$$f(\sigma,c) = \sqrt{J_2} + \alpha\, I_1 - k(c)$$

The plastic potential has the same form :

$$g(\sigma) = \sqrt{J_2} + \beta\, I_1 + \text{constante}$$

where α, β and k are deduced from c, φ and ψ by :

$$\alpha = \frac{2\sin\varphi}{\sqrt{3}(3-\sin\varphi)} \qquad \beta = \frac{2\sin\psi}{\sqrt{3}(3-\sin\psi)} \qquad k = \frac{6c\cos\varphi}{\sqrt{3}(3-\sin\varphi)}$$

Angles φ and ψ are constants ; parameter c is the hardening parameter. Its evolution is described by:

$$\dot{c} = -\dot{\lambda}\frac{\mu}{1+\chi}$$

where $\mu = E / [2(1+\nu)]$ is the shear modulus.

The hardening can be positive or negative according to the value of $1+\chi$.

The hardening law and the flow rule make it possible to establish a relation between the evolutions of c and the volumetric plastic strain.

Parameters for the Drucker Prager with hardening model (IMOD=14) :

- density (RO) [kg m^{-3}]
- Young's modulus (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- initial value of the hardening parameter c (C) [Pa]
- friction angle φ (PHI) [deg] and dilatancy angle ψ (PSI) [deg]
- coefficient χ (XHI) of the hardening law

3.2.11. Parabolic criterion (IMOD=15)

The parabolic criterion allows modelling the strength of concrete in a schematic way.

It is an elastic-perfectly plastic model.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The parabolic plasticity has been developed to represent the behaviour of concrete. It is defined by the formula:

$$f(\sigma) = J_2 + (R_c - R_t)\, I_1 / 3 - R_c R_t / 3 = 0$$

Parameter Rc corresponds to the simple compressive strength and Rt is the tensile strength of the material. The tensile strength of concrete Rt is considered equal to zero or equal to a few MPa. The compressive strength Rc can vary depending on the cement composition and the formulation: common values range between 25 and 50 MPa (Fauchart, 1977).

The flow rule is associated.

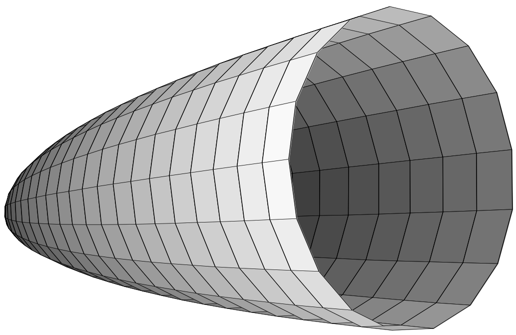


Figure 5 – Visualization of the yield surface in the principal stresses space

Parameters for the parabolic criterion (IMOD=15) :

- density (`RO`) [kg m^{-3}]
- Young's modulus (`YOUNG`) [Pa]
- Poisson's ratio ν (`POISS`) [-]
- compression strength Rc (`RC`) [Pa]
- tensile strength Rt (`RT`) [Pa]

3.2.12. Modèle de Vermeer (IMOD=16)

The constitutive model proposed by Peter Vermeer (1978, 1980, 1982) is an elastoplastic law with two hardening mechanisms, designed to reflect the behaviour of sands. The first mechanism is a purely volumetric mechanism (consolidation mechanism) and the second mechanics is purely deviatoric (shear mechanism), based on the failure criterion by Matsuoka and Nakai (1974).

The elasticity of the law proposed by Peter Vermeer corresponds to a secant shear modulus depending on the stress state and a zero Poisson's ratio:

$$\sigma_{ij} = 2\, G_s \varepsilon_{ij}$$

$$\text{avec } G_s = G_o\, [\sigma_n / p_o]^{(1-\beta)} \text{ and } G_o = 3p_o/(2\varepsilon_o^e)$$

where p_o, ε_o^e, β are three constants and σ_n is the invariant of the stress tensor defined by $\sigma_n^2 = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)/3$

Description of the model as implemented in CESAR:

It is an elastoplastic model with two hardening plastic mechanisms.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

Volumetric plastic mechanicsm:

The criterion is given by :

$$f_v(\sigma, \varepsilon^p_{vc}) = \varepsilon_o^c\, [\sigma_n / p_o]^{\beta} - \varepsilon^p_{vc}$$

where ε_o^c is a constant and ε^p_{vc} is the hardening parameter. The flow rule for this mechanism is associated.

Deviatoric plastic mechanicsm:

The second yield surface is defined by another criterion:

$$f_c(\sigma,x) = 3\, p\, J_2 - I_3\, A(x)$$

where A(x) is a scalar function defined by:

$$A(x) = \frac{27\,(3 + h(x))}{(2\,h(x) + 3)\,(3 - h(x))} \qquad c = \frac{6 \sin \varphi_p}{3 - \sin \varphi_p}$$

$$h(x) = \sqrt{\frac{x^2}{4} + cx} - \frac{x}{2} \qquad x = \gamma^p\; 2\,\frac{G_o}{p_o}\left(\frac{p_o}{\sigma_n}\right)^{\beta}$$

In these expressions, φ_p represent the peak friction angle, and γ^p the plastic distorsion defined by $(\gamma^p)^2 = \frac{1}{2}\, \varepsilon_{ij}^p \varepsilon_{ij}^p$

The flow rulefor this mechanism is non-associated. The plastic potential is given by:

$$g_c(x) = \sqrt{\frac{2}{3} s_{ij}\, s_{ij}} - \frac{4}{3}\, p \sin \psi_m$$

The dilatancy angle ψ_m is connected to the stress state through the relation proposed by Rowe (1971) :

$$\sin \psi_m = \frac{\sin \varphi_m - \sin \varphi_{cv}}{1 - \sin \varphi_m \sin \varphi_{cv}}$$

where φ_{cv} is the friction angle at constant volume, assumed to characterise the ultimate shear state of the material. The angle φ_m is the mobilised friction angle, related to the stress state by the relation:

$$\sin \varphi_m = \frac{3q}{6p+q}$$

The parameters of the Vermeer law are determined from axisymmetrical drained compression triaxial tests with an unloading phase. This determination is discussed for instance by Mestat (1993).

Parameters for the Vermeer model (IMOD=16) :

- density (RO) [kg m^{-3}] ;
- Young's modulus (YOUNG) [Pa] ;
- Poisson's ratio ν (POISS) [-] ;
- parameter ε_o^e (EPS0) ;
- friction angleφ_{cv} for which shear takes place at constant volume (PHICV) ;
- peak friction φ_p (PHIP) ;
- parameter β (BETA) ;
- parameter ε_o^c (EPSC0) ;
- reference pressure po (P0) .

Note : the elastic part of the model has been replaced by a linear model, but parameters EPS0 and BETA

Are involved in the hardening law of the first mechanism.

3.2.13. Nova model (IMOD=17)

The Nova model (1982) is an elastoplastic model with isotropic hardening, inspired by the models of the Cam-Clay family (see IMOD=18 below), but adapted to the description of the sand behaviour. It has been developed on the basis of experimental results on cylindrical samples, which explains its formulation in term of the stress invariant p (mean pressure) and q (deviatoric stress) and of the plastic strain invariants ε_v^p (volumetric plastic strain) and ε_d^p (deviatoric plastic strain).

The elasticity proposed by Nova is isotropic and non-linear:

$$\eta_{ij} = \frac{s_{ij}}{p} \qquad d\varepsilon_{ij} = L_o \, d\eta_{ij} + B_o \frac{dp}{3p}\delta_{ij}$$

where Bo and Lo are two parameters and ▪ij are the components of the Kronecker delta.

One of the fundament of this law is the stress-dilatancy relation adopted by R. Nova depending on the value of the stress ratio q/p.

Description of the model as implemented in CESAR:

It is an elastoplastic model characterised by two flow regimes, which can be distinguished by the value of the stress ratio q/p. In the two regimes, there is a hardening of the failure surface. One of the regimes (q/p<M/2) is associated, while the other (q/p>M/2) is not associated.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The stress-dilatancy relation proposed by R. Nova is different depending if the stress ratio q/p is larger or smaller than a value notated M/2.

- if q/p < M/2,

The flow rule is associated and the expression of the yield surface is provided by:

$$f(p,q,p_c) = \frac{4\mu \, q^2}{M^2 p^2} + 1 - \frac{p_c^2}{p^2}$$

where μ is a constant and p_c is the hardening parameter.

- if q/p > M/2,

The flow rule is non-longer associated. The yield surface is the one proposed by Tatsuoka and Ishihara (1974):

$$f(p, q, p_c) = \frac{q}{p} - M + m \ \ln\left(\frac{p}{p_u}\right)$$

where the parameter p_u is directly related to the hardening parameter:

$$p_u = \frac{p_c}{\sqrt{1+\mu}} \exp\left[-\frac{M}{2m}\right]$$

The plastic potential is given by:

$$G(p, q) = q/p - M/(1-\mu) \left[1 - \mu \, (p/p_{cg})^{(1-\mu)/\mu} \right]$$

where p_{cg} is a constant that is usually not necessary to specify. The shape of the potential allows ensuring that the volumetric and deviatoric plastic strains satisfy:

$$d\varepsilon_v^p = \frac{M-(q/p)}{\mu} d\varepsilon_d^p$$

The hardening parameter p_c follows an evolution rule similar to the one of Cam-Clay. The difference comes from the consideration of the deviatoric term ε_d^p in the Nova law:

$$p_c = p_c^\circ \ \exp\left[- \frac{\varepsilon_v^p + D\varepsilon_d^p}{l - B_o} \right]$$

The values of the parameters of the Nova model for various sands can be found in Mestat (1993), as well as the recommendations for the determination of these parameters from compression triaxial tests, axisymmetrical and drained with an unloading phase.

Parameters for the Nova model (IMOD=17) :

- density (`RO`) [kg m^{-3}] ;
- Young's modulus (`YOUNG`) [Pa] ;
- Poisson's ratio ν (`POISS`) [-] ;
- parameter L_o for the non linear elasticity (`L0`) [-] ;

(ignored, since the model was implemented with a linear elasticity) ;

- parameter B_o of the elastic model (`B0`) [-], also involved in the hardening law ;
- parameter defining the transition between both flow regimes M (`M`) [-] ;
- parameters l (`L`) [-] and D (`D`) [-] of the hardening law ;
- parameters m (`MM`) [-] and μ (`MU`) [-] ;
- initial value p_c° (`PC0`) [Pa] of the hardening parameter p_c .

3.2.14. Modified Cam-Clay model (IMOD=18)

The term Cam-Clay refers to a family of constitutive models developed in the 1960s by the group of soil mechanics of the University of Cambridge (Roscoe et al., 198, Schofield and Wroth, 1968). They are elastoplastic models with hardening, essentially dedicated to modelling the behaviour of clay remoulded in laboratory. The original formulation uses a non-linear isotropic elasticity defined by:

$$d\varepsilon_v^e = -\frac{\kappa}{1+e_o}\frac{dp}{p} \quad d\varepsilon_d^e = \frac{dq}{3G}$$

In CESAR-LCPC, the model IMOD=18 adopts the yield surface of the model called “Modified Cam-Clay” and the corresponding hardening law. However, the available model does not adopt the original elastic law, which is replaced by linear isotropic elasticity.

Description of the model as implemented in CESAR:

It is an elastoplastic model with isotropic hardening.

The elastic part of the model is linear and isotropic, defined by the input of Young’s modulus E and Poisson’s ratio ν.

The plasticity criterion is given by:

$$f = q^2 + M^2 p (p - p_c) = 0$$

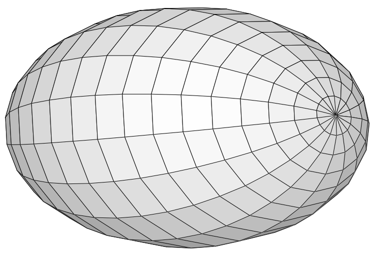


Figure 6 - Visualization of the yield surface for the Cam-Clay in the principal stresses space

The parameter M is fixed. It controls the eccentricity of the ellipse that defines the boundary of the elastic domain in the (p, q) space. It also represents the slope of the critical state line in the same plane. It can be related to the internal friction angle φ' by:

$$M = \frac{6 \sin \varphi'}{3 - \sin \varphi'}$$

The parameter p_c is the hardening parameter.

The flow rule is associated.

The hardening rule relates the variations of the hardening parameter with the volumetric plastic strain ε_v^p by:

$$p_c = p_{co} \exp(\alpha_p \varepsilon_v^p) \qquad \text{with } \alpha_p = \frac{1+e_o}{\lambda - \kappa}$$

where p_c corresponds to the consolidation pressure and e_o to the initial void ratio associated with the initial consolidation pressure p_{co}.

The parameter λ corresponds to the slope of the initial consolidation curve in a (e, ln p) diagram representing the results of an isotropic compression test. The parameter κ corresponds to the slope of the unloading-reloading curves in the same diagram. It is generally admitted that λ and κ are generally related to the coefficients C_c and C_s that are deduced from classical oedometer tests through:

$\lambda = C_c/\ln 10$ et $\kappa = C_s/\ln 10$).

Parameters for the Modified Cam-Clay model (IMOD=18) :

• density (RO) [kg m^{-3}] ;

• Young's modulus (YOUNG) [Pa] ;

• Poisson's ratio ν (POISS) [-] ;

• slope of the initial consolidation curve in the e-ln p space for an isotropic compression test λ (ALOE) [-];

• slope of the unloading-reloading curves in the same e - ln p diagram κ (AKOE) [-];

• slope of the critical state line M (AMC) [-] ;

• initial void ratio e_o (OED) [-] ;

• initial consolidation pressure p_{co} (PC0) [Pa] ;

3.2.15. Prévost and Hoeg model (IMOD=19)

This constitutive model has been developed to represent the softening in soils and rocks, i.e. the progressive strength loss after the stress peak. It is based on the von Mises criterion with a specific hardening. The hardening variable is the plastic shear strain. The initial elastic domain is void, and the criterion does not depend on the mean stress.

It is an elastoplastic model with hardening.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The pplasticity criterion is the same as for the von Mises model:

$$f(\sigma) = J_2 - k^2 = \frac{1}{2} s_{ij}\, s_{ij} - k^2$$

The flow rule is associated.

The evolution of the hardening parameter k depends on that of the plastic strains:

$$k = \frac{A\,[B\,(\bar{\varepsilon}_d^{\,p})^2 + \bar{\varepsilon}_d^{\,p}]}{1 + (\bar{\varepsilon}_d^{\,p})^2} / \sqrt{3}$$

where $\bar{\varepsilon}_d^{\,p}$ is defined by : $d\bar{\varepsilon}_d^{\,p} = 10^3 \times [\frac{2}{3} (d\varepsilon_d^{\,p})_{ij}\, (d\varepsilon_d^{\,p})_{ij}\,]^{1/2}$

$\varepsilon_d^{\,p}$ denoting the deviatoric part of the plastic strain tensor.

The model involves 4 parameters: E, ν, A, B. The meaning of A and B is illustrated by the figure below. The hardening is at first positive, then negative.

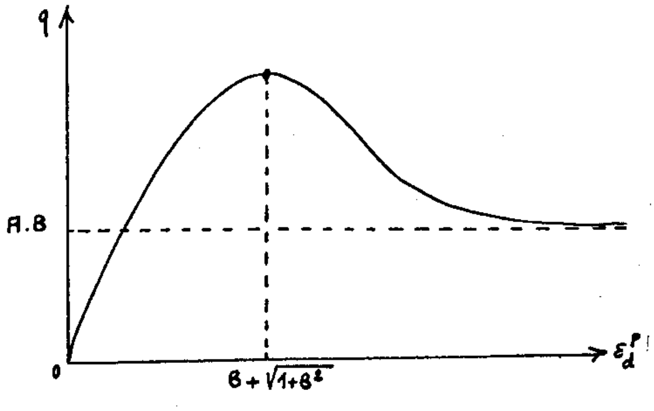


Figure 7 – Parameters for the Prévost and Hoeg model

Parameters for the Prévost and Hoëg model (IMOD=19) :

- density (`RO`) [kg m^{-3}] ;
- Young's modulus (`YOUNG`) [Pa] ;
- Poisson's ratio ν (`POISS`) [-] ;
- coefficients A (`A0`) [-] and B (`B0`) [-].

3.2.16. Directional model (IMOD=20)

Contrary to the criteria defined based on the stress invariants (the value of which depend on the orientation of principal stresses), this criterion allows taking into account the anisotropic character of soil or rock layers. The principle is to define for the soil continuum a direction of plastic strain modelling the role of discontinuities of a fractured continuum. This model is inspired by the law proposed by Cramer to model an interface, or discontinuity, by a thin band of continuous material. The criterion has been extended by Frank et al (1982).

It is an elastic-perfectly plastic model.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion imposes a Coulomb strength condition on the normal stress σ_n and on the tangential stress τ acting on a surface perpendicular to a given unit vector $\underline{n}$:

$$f(\sigma) = |\tau| + \sigma_n \tan \tau - c$$

where c is the cohesion and φ the friction angle.

The stresses τ and σ_n are given by:

$$\sigma_n = (\underline{\underline{\sigma}} . \underline{n}) . \underline{n} \qquad \tau = \underline{\underline{\sigma}} . \underline{n} - \sigma_n . \underline{n}$$

In bidimensional condition, the unit vector n is defined by the angle between the horizontal and the plane of the facet

$$\underline{n} = -\sin \alpha \underline{e}_x + \cos \alpha \underline{e}_y$$

which leads to:

$$\sigma_n = (\sigma_{xx} + \sigma_{yy})/2 - (\sigma_{xx} - \sigma_{yy}) (\cos 2\alpha)/2 - \sigma_{xy} \sin 2\alpha$$

$$\tau = - (\sigma_{xx} - \sigma_{yy}) (\sin 2\alpha)/2 + \sigma_{xy} \cos 2\alpha$$

In three-dimensional condition, the user inputs the three components of the unit vector $\underline{n}$.

Le plastic potential has the same form as the criterion :

$$g(\sigma) = |\tau| + \sigma_n \tan \psi + \text{constant}$$

where ψ is an angle accounting for dilatancy.

Parameters for the direction criterion (IMOD=20) :

- density (`RO`) [kg m^{-3}] ;
- Young's modulus (`YOUNG`) [Pa] ;
- Poisson's ratio ν (`POISS`) [-];
- cohrsion c (`C`) [Pa]
- friction angleφ (`PHI`) [deg] and dilatancy angle ψ (`PSI`) [deg] ;
- in bidimensional condition, angle α (`ALPHA`) [deg].
- in three-dimensional condition, three compoments u1, u2, u3 (`U1, U2, U3`) [-] of the unit vector $\underline{n}$

3.2.17. Hoek and Brown model (IMOD=24)

For rocks, Hoek and Brown have proposed a failure criterion in which the maximum shear stress applied on a face increases less steadily with the normal stress p than with the usual Mohr-Coulomb criterion.

It is an elastic-perfectly plastic model.

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion is given by:

$$f(\sigma) = \Sigma_1 - \Sigma_3 - \sigma_u\sqrt{m\frac{\Sigma_3}{\sigma_u} + s}$$

where m is a shape parameter, s is the c and σ_u is a strength parameter. It can easily be checked that $\sigma_u\sqrt{s}$ gives the simple compression strength of the material

The flow rule in CESAR is associated.

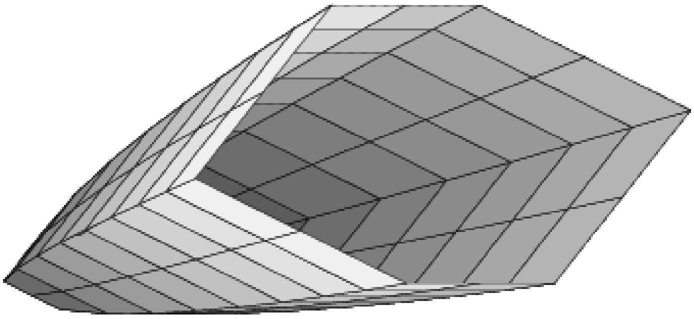


Figure 8 - Visualization of the yield surface for the Hoek and Brown in the principal stresses space

Parameters for the Hoek and Brown model (IMOD=24) :

- density (`RO`) [kg m^{-3}] ;
- Young's modulus (`YOUNG`) [Pa] ;
- Poisson's ratio ν (`POISS`) [-] ;
- strength parameter σ_u (`SU`) [Pa] ;
- fracture coefficient s (`S`) [-] ;
- shape parameter m (`M`) [-].

3.2.18. Mélanie models (IMOD=34 in 2D and 35 in 3D)

It is an anisotropic elastoplastic constitutive law with hardening which is inspired by the law of the University of Cambridge, by the studies carried out at Université Laval de Quebec under the supervision of F. Tavenas and S. Leroueil and by works carried out at the LCPC on the behaviour of natural soft clays.

Melanie model is an elastoplastic model with hardening.

The elastic part of the model is linear and transversely isotropic, defined by the same parameters as the linear transversely isotropic model IMOD=2 (see 0).

The yield surface has been identified on the basis of triaxial tests interpreted in terms of the "reduced" stresses s' and t defined by:

$t = \sigma_1 - \sigma_3 \qquad\qquad s' = (\sigma_1 + \sigma_3)/2$

where σ_1 and σ_3 are defined in the context of triaxial tests: σ_1 is the compression axial stress and σ_3 is the lateral confining stress. The expression of the criterion is:

$$f(s', t, s'_p) = A^2 (s' \cos\beta + t \sin\beta - s'_p / A C)^2 + B^2 (t \cos\beta - s' \sin\beta)^2 - s_p'^2 / C^2$$

with

$$s'_p = 0{,}3 \ (1 + K_o) \ \sigma'_p \qquad\qquad A = 2 \ (\sin\beta + \cos\beta)$$

$$B^2 = A^2 \cos^2\beta \ (2/A - C\cos\beta) / (C \sin^2\beta) \qquad\qquad C = 0{,}6$$

The elastic domain in the (s',t) plane is, limited by an ellipse the major axis of which makes an angle β with the isotropic compression axis Os' determined by the relation:

$$\tan\beta = (1-K_o)/(1+K_o)$$

where K_o denotes the normally consolidated coefficient of earth pressure. Parameter s'_p is the hardening parameter.

The direction of the plastic strain increment is defined by:

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial f/\partial \sigma_{ij}}{|\partial f/\partial \sigma_{ij}|} + \eta \frac{OM}{|OM|}$$

If $\eta=0$ the flow rule is associated ; si $\eta=1$ it is non-associated.

The hardening law must be defined, that is describing the evolution of parameter s'_p. Based on the curve describing the evolution of the void ratio as a function of s'_p:

$$e = e_{\lambda o} - \lambda \ln (s'_p/s'_1)$$

The relation between the void index and the volumetric strain is then used:

$$de = (1+e_o)\, d\varepsilon_v$$

to get:

$$d\varepsilon_v^p = \frac{de}{(1+e_o)} - d\varepsilon_v^e = -\lambda \frac{ds'_p}{(1+e_o)\, s'_p} - d\varepsilon_v^e$$

Eventually the elastic volumetric strain $d\varepsilon_v^e$ connected to ds'_p by :

$$d\varepsilon_v^e = \frac{\alpha}{1+e_o} ds'_p$$

with

$$\frac{\alpha}{1+e_o} = \frac{(1-\nu_h-2n\,\nu_v^2)}{E_v} \cdot \frac{(1+\nu_h)(2n\nu_v-1+\nu_h)-n\,(1-n\nu_v^2)}{n^2\nu_v^2(1+\nu_h)-n(1-n\nu_v^2)(1-\nu_h)}$$

where n is the ratio E_h/E_v.

See Lépidas and Magnan (1990) or Mestat (1993° to obtain precisions on the determination of the model parameters.

Parameters for the Melanie model (IMOD=34 in 2D and 35 in 3D) :

- density(RO) [kg m^{-3}] ;
- Young's modulus in the isotropic plane E_h (E1) [Pa] ;
- Young's modulus in the direction of the symmetry axis E_v (E2) [Pa] ;
- Poisson's ratios in the isotropic plane ν_h (P1) [-] and transverse ν_v (P2) [-] ;
- shear modulus G (G2) [Pa] ;
- in bidimensional condition : angle θ between Ox and the isotropic plane (TETA) [deg] ;
- in three-dimensional condition :

Two angles θ and φ defining the direction of the axisymmetry axis (TETA, PHI) [deg];

- slope λ of the initial consolidation curve (diagram e –ln s'_p) (ALPHA) [-] ;
- initial void ratio e_o (OED) [-] ;
- initial effective vertical stress (SIVO) [Pa] ;
- initial preconsolidation pressure (initial value of $\sigma'_p = s'_p/0{,}6$) (SIPO) [Pa] ;
- K_{sc} : overconsolidated coefficient of earth pressure (used for the initialization of the hardening parameter) (CPSC) [-] ;
- K_{nc} : normally consolidated coefficient of earth pressure (= K_o defining the orientation of the ellipse in the (s',t) plane) (CPNC) [-] ;
- normality index η (JTA) [-] ;
- tolerance for the internal iterations for the computation of $e_{\lambda o}$ (TOLC) [-].

3.2.19. Anisotropic Tresca model (IMOD=40)

This model is only available in plane strain.

In this model, the ultimate shear stress depends on the direction of the larger compressive principal stress for a purely coherent soil. The expression of the anisotropic strength in function of the orientation of the principal stresses is approximated by:

$$C_u^{\,2}(\beta) = C_{uv}^{\,2}\cos^2\beta + C_{uh}^{\,2}\sin^2\beta$$

where β is the inclination of the major principal stress with the axis of anisotropy, C_{uv} the undrained cohesion on a compression test on a vertical sample ; C_{uh} the undrained cohesion on a compression test on a horizontal sample.

The angle β is defined by the relation (in plane strain conditions):

$$\tan(2\beta) = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

The model IMOD=40 is a perfectly plastic elastoplastic model.

The elastic part of the model is assumed linear orthotropic of revolution. It is described by the same parameters as in the model IMOD=2.

In plane strain conditions, the criterion writes:

$$f(\sigma) = \sigma_1 - \sigma_3 - 2\,C_u(\beta) \le 0$$

The flow rule is associated.

Note: the undrained cohesions C_{uv} et C_{uh} are assume to vary as a function of the depth in a linear way:

$C_{uv} = C_{uv}(0) + a_v z$ $C_{uh} = C_{uh}(0) + a_h z$

Parameters for the anisotropic Tresca model (IMOD=40) :

- density(RO) [kg m^{-3}] ;
- Young's modulus in the isotropic plane E_h (E1) [Pa] ;
- Young's modulus in the direction of the symmetry axis E_v (E2) [Pa] ;
- Poisson's ratios in the isotropic plane ν_h (P1) [-] and transverse ν_v (P2) [-] ;
- shear modulus G (G2) [Pa] ;
- angle θ between Ox and the isotropic plane (TETA) [deg] ;
- vertical cohesion $C_{uv}(0)$ for z = 0 (A1) [Pa] ;
- vertical gradient of vertical cohesion a_v (B1) [Pa . m^{-1}] ;
- horizontal cohesion $C_{uh}(0)$ for z = 0 (A2) [Pa] ;
- vertical gradient of horizontal cohesion a_h (B2) [Pa . m^{-1}].

3.2.20. de Buhan and Sudret model for reinforced materials (IMOD=43)

This model has been developed by de Buhan and Sudret (2000) for the modelling of grounds reinforced by linear inclusions (nails, bolts, micropiles etc…). The principle is to replace the heterogeneous continuum made up of the soil and the inclusions by an equivalent homogenous continuum. The inclusions introduce an additional stiffness in tension-compression in a given direction; no flexural stiffness of the inclusions is taken into account.

An interesting feature of the model consists in the fact that plasticity is treated separately in the soil and in the inclusions, which leads to a global hardening behaviour. Model 43 corresponds to the combination of Drucker-Prager model without hardening for the soil (see IMOD=13 above) and of a perfectly plastic elastoplastic model for the inclusions (in which the stress state is characterised by a single scalar). Will be therefore provided on the one hand the data describing the soil behaviour before reinforcement, and on the other hand the geometrical and mechanical properties of the bolts and their arrangement in space. The solver makes it possible to define two sets of reinforcement inclusions oriented in different directions or having different mechanical properties. The number of sets is provided by the NRENF parameter (equal to 0, 1 or 2).

For each inclusion set, the user must provide the Young's modulus of the constitutive material of the inclusions (K), the cross-sectional area of the inclusion (SA), the tensile strength (S) and the ratio between the tensile and the compression strength (ETA). The other parameters are related to the geometrical arrangement of the inclusions. The software allows taking into account four geometrical arrangements, the chosen arrangement being defined by the parameter TYP:

- Homogenous reinforcement (TYP=0): the inclusions are parallel to a given direction.

In plane strain conditions or in plane stress conditions, this direction is defined by the angle ALPHA between the horizontal axis Ox and the inclusions.

In axisymmetrical strain, the common direction is necessarily the axis parallel to the revolution axis and the single parameter to be provided is the volumetric fraction FV.

- In three-dimensional condition, the direction of the inclusions is defined by two angles A and B (in the interval [-90 ;90°]). Angle A is the angle between Ox and the projection of the reinforcement direction onto the horizontal plane (z=0); B is the angle between the horizontal plane and the reinforcement direction. A unit vector of the reinforcement direction is given by : (cos A cos B, sin A cos B, sin B).

The density of the reinforcement is characterised by the volumetric fraction of the inclusions FV, obtained by multiplying the number of inclusions per squared meters by the cross-sectional area of an inclusion (or by computing the sum of the cross-sectional areas of the inclusions installed in a given surface perpendicular to their direction and divided by the area of this surface).

- Radial reinforcement (TYP=1): the inclusions are placed in planes perpendicular to a given line Δ. In each plane they converge towards the intersection of the plane with Δ:

In plane strain conditions or in plane stress condition, the line Δ is perpendicular to the mesh plane and the coordinates X_C and Y_C of the convergence point must be provided. The density of the reinforcement system varies as the inverse of the distance to this point. The user must provide the volumetric fraction of the inclusions at a unit distance from the centre (X_C, Y_C) of the system FVR1.

In axisymmetrical strain conditions, the line Δ is the axis of revolution symmetry and the user must only provide the volumetric fraction at a unit distance from the centre of the system FVR1.

- In three-dimensional condition, the line Δ is defined by one point and its direction. The user provides the coordinates X, Y, Z of one point of Δ and the direction of Δ is defined by two angles A and B (in the interval [-90 ;90°]) Angle A is the angle between Ox and the projection of Δ onto the horizontal plane (z=0); B is the angle between the horizontal plane and Δ. A unit vector of the reinforcement direction is given by: (cos A cos B, sin A cos B, sin B). The user must also provide the volumetric fraction at a unit distance from the centre of the system FVR1.

- Cylindrically diverging reinforcement (TYP=2, only available in axisymmetrical or 3D conditions): the inclusions are placed along cones of axis Δ and at a given angle from the axis.

In axisymmetrical conditions, the axis of the cones Δ is the revolution axis. The angle BETA between the symmetry axis Oz and the reinforcement direction and the volumetric fraction FVR1 at the unit distance from the symmetry axis of revolution must be provided.

In three-dimensional condition, the axis Δ of the cones is defined by the coordinates (X, Y, Z) of one of its points, and two angles A and B in the interval [-90 ;90°] defining the orientation of Δ. Angle A is the angle between Ox and the projection of Δ onto the horizontal plane (z=0); B is the angle between the horizontal plane and Δ. A unit vector of the reinforcement direction is given by: (cos A cos B, sin A cos B, sin B). The user must also provide the volumetric fraction FVR1 of the inclusions at a unit distance from the symmetry axis.

- The spherically diverging reinforcement (TYP=3, only available in axisymmetrical or 3D conditions): the inclusions are placed along lines converging in one point. The density of the reinforcement system varies as the inverse of the squared distance to this point.

In axisymmetrical conditions, the convergence point is on the revolution symmetry axis. Its vertical coordinate Z_C and the volumetric fraction FVR1 at a unit distance from this point must be provided.

- In three-dimensional condition, the user provides the coordinates (X_C, Y_C, Z_C) of the convergence point, and the volumetric fraction FVR1 at a unit distance from this point.

In bidimensional conditions, the model comprises 21 parameters listed below.

Parameters for the de Buhan and Sudret model (IMOD=43) in bidimensional condition

Parameters for the ground behaviour with no inclusions (Drucker-Prager model) :

- density (RO) [kg m^{-3}] ;
- Young's modulus (YOUNG) [Pa] ;
- Poisson's ratio ν (POISS) [-];
- cohesion c (C) [Pa]
- friction angleφ (PHI) [deg] ;
- dilatancy angle ψ (PSI) [deg].

Parameters describing the reinforcement scheme:

- number of inclusions sets (NRENF) ;
- parameters for the first set of inclusions TYP1, K1, SA1, S1, ETA1, X1, Y1, FV1 ;
- parameters for the second set of inclusions TYP2, K2, SA2, S2, ETA2, X2, Y2, FV2.

Parameters Ki, SAi, Si and ETAi are respectively, for the inclusion set number i, the Young's modulus of the constitutive material of the inclusions, the cross-sectional area of one inclusion, the tensile strength and the ratio of the compression strength to the tensile strength.

The correspondance between parameters Xi, Yi and FVi with those descrived above depends on the value of the indicator TYP (TYP1 or TYP2) and on the indicator INAT (=1 in plane strain, =2 in axisymmetrical condition, =3 in plane stress) and is given un the following tables, where the notation * indicates that a parameter is not used for the values of INAT and TYP :

	TYP	X	Y	FV
INAT=1 or 3 (plane strain / plane stress)	0	ALPHA	*	FV volumetric fraction
	1	XC	YC	FVR1 volumetric fraction at a unit distance from the symmetry axis
	2	Values of TYP unavailable if INAT=1 or 3		
	3			

INAT =2 calcul axisymétrique	0	*	*	FV volumetric fraction
	1	*	*	FVR1 volumetric fraction at a unit distance from the symmetry axis
	2	BETA	*	FVR1 volumetric fraction at a unit distance from the symmetry axis
	3	ZC	*	FVR1 volumetric fraction at a unit distance from the convergence point

For three-dimensional computations, the model comprises 31 parameters listed below.

Parameters for the de Buhan and Sudret model (IMOD=43) in three-dimensional condition

Parameters for the ground behaviour with no inclusions (Drucker-Prager model) :

- density (RO) [kg m^{-3}] ;
- Young's modulus (YOUNG) [Pa] ;
- Poisson's ratio ν (POISS) [-];
- cohesion c (C) [Pa]
- friction angle φ (PHI) [deg] ;
- dilatancy angle ψ (PSI) [deg].

Parameters describing the reinforcement scheme:

- number of inclusions sets (NRENF) ;
- parameters for the first set of inclusions TYP1, K1, SA1, S1, ETA1, X1, Y1, Z1, A1, B1, C1, FV1 ;
- parameters for the second set of inclusions TYP2, K2, SA2, S2, ETA2, X2, Y2, Z2, A2, B2, C2, FV2.

Parameters Ki, SAi, Si and ETAi are respectively, for the inclusion set number i, the Young's modulus of the constitutive material of the inclusions, the cross-sectional area of one inclusion, the tensile strength and the ratio of the compression strength to the tensile strength.

The correspondance between parameters Xi, Yi, Zi, Ai, Bi, Ci and and FVi with those descrived above depends on the value of the indicator TYP (TYP1 or TYP2) and is given un the following tables, where the notation * indicates that a parameter is not used for the values of INAT and TYP :

TYP	A	B	C	X	Y	Z	FV
0	A1	B1	*	*	*	*	FV volumetric fraction
1	A1	B1	*	X1	Y1	Z1	FVR1 volumetric fraction at a unit distance from the symmetry axis
2	A1	B1	C1	X1	Y1	Z1	FVR1 volumetric fraction at a unit distance from the cone axis
3	*	*	*	X1	Y1	Z1	FVR1 volumetric fraction at a unit distance from the symmetry axis

3.2.21. Standard Willam-Warnke model (IMOD=47)

The Willam-Warnke model is considered as a criterion of Drucker-Prager type model adapted to concrete in the domain of small confining pressures. Compared to the Drucker-Prager criterion, the criterion depends in addition on the Lode angle θ. This model corresponds to two different regimes depending on the value of the κ parameter: if κ=0, the model is perfectly plastic and if $\kappa \neq 0$, the model includes hardening.

Perfectly plastic behaviour (κ=0):

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion is given by:

$$f(\sigma) = \sqrt{J_2} + f(\theta)\,(\sigma-\rho_o)$$

where σ corresponds to the trace of the stress tensor and where the friction term $f(\theta)$ is provided by:

$$f(\theta) = \frac{u+v}{w} \text{ with } \begin{cases} u=2\, f_c\,(f_c^2-f_t^2)\cos\theta \\ v= f_c\,(2\, f_c-f_t)\sqrt{4\,(f_c^2-f_t^2)\cos^2\theta+5\, f_t^2-4f_c\, f_t} \\ w=4\,(f_c^2-f_t^2)\cos^2\theta + (f_c-2\, f_t)^2 \end{cases}$$

$f_c = f\,(60°)$: friction term corresponding to the compression meridian

$f_t = f\,(0°)$: friction term corresponding to the tension meridian

The coefficient ρ_o, f_c et f_t are computed in the code as functions of the values of the direct compressive strength σ, of the direct tensile strength σ_t and of the biaxial compressive strength σ_{bc}.

The flow rule is associated.

Note: if κ=0 the values of parameters A_o and B_o are ignored.

Model with hardening ($\kappa \neq 0$)

The plasticity criterion is given by:

$$f(\sigma, z) = \tau + f(\theta)\,(\sigma-z\rho_o)$$

In this case the model uses four parameters: ρ_o, f_c, f_t and z.

The flow rule is associated :

$$d\varepsilon^p = d\lambda\, [\frac{s}{2\tau} + \frac{\delta}{3}\,\mathbf{1}]$$

The hardening parameter z depends on the hardening variable γ_{eq}^p through:

$$z = A_0 - (A_0-B_0)\,(1-\exp\,[-\kappa\gamma_{eq}^p])$$

and the variations $d\gamma_{eq}^p$ of γ_{eq}^p are described by by $d\gamma_{eq}^p = d\lambda$.

Note 1: if A0>B0, hardening is positive and negative otherwise.

Note 2: the model does not allow taking into account the contracting plastic behaviour. To this aim, the modified Willam-Warnke model will be used.

Parameters for the standard Willam-Warnke model (IMOD=47) :

- density (RO) [kg m^{-3}] ;

- Young's modulus (YOUNG) [Pa] ;
- Poisson's ratio ν (POISS) [-] ;
- compressive strength, σ_c (FC) [Pa] ;
- tensile strength, σ_t (FT) [Pa] ;
- Strength in biaxial compression, σ_{bc}(FBC) [Pa] ;
- (relative) initial elastic limit A_o(A0) [-] ;
- (relative) final elastic limit B_o(B0) [-] ;
- hardening factor κ (KAPPA) [-].

3.2.22. Modified Willam-Warnke model (IMOD=48)

The modification of this criterion consists in taking into account in a more realistic manner the curved shape of the meridians. It is used to model the hardening behaviour (H>0) and the softening behaviour of concrete (H<0) based on a initial elastic domain that includes the failure points in direct compression, in biaxial compression and in direct tension. It allows taking into account the contracting plastic behaviour in the domain of significant hydrostatic pressures and the dilatancy behaviour in the domain of small hydrostatic pressures.

This model is implemented in the perfectly plastic case and in the hardening case.

Perfectly plastic behaviour (κ=0):

The elastic part of the model is linear and isotropic, defined by the input of Young's modulus E and Poisson's ratio ν.

The plasticity criterion is given by :

$$f(\sigma) = \tau + f(\theta)\frac{(\sigma - \rho_o)\ (\sigma + 2p_{cr} - \rho_o)}{2p_{cr}}$$

where p_{cr} is a strictly positive constant in the case of perfect plasticity.

The coefficients ρ_o, p_{cr}, f_c and f_t are computed in code as functions of the values of the direct compressive strength σ_c, of the direct tensile strength σ_t, of the biaxial compressive strength σ_{bc} and of the result of a triaxial test, characterised by the three principal stresses at failure S_1, S_2 et S_3 (necessarily $S_2=S_1$ or $S_2=S_3$).

Hardening behaviour ($\kappa \neq 0$):

The failure criterion is defined by the same expression:

$$f(\sigma) = \tau + f(\theta)\ \frac{(\sigma - \rho_o)\ (\sigma + 2p_{cr} - \rho_o)}{2p_{cr}}$$

but p_{cr} may vary.

The flow rule is associated.

The hardening parameter z is related to the hardening variable d'écrouissage $\gamma_{eq}{}^p$ by:

$$p_{cr} = p_{cr}{}^{\circ}\ \{ A_0 - (A_0\text{-}B_0)\ (1\text{-}\exp\ [-\kappa\gamma_{eq}{}^p])\ \}$$

where variations of γ_{eq}^p are described by $d\gamma_{eq}^p = d\lambda$.

Note : if $A_0 > B_0$, hardening is positive, otherwise it is negative.

Parameters for the modified Willam Warnke model (IMOD=48) :

- density (RO) [kg m^{-3}] ;
- Young's modulus (YOUNG) [Pa] ;
- Poisson's ratio ν (POISS) [-] ;
- compressive strength, σ_c (FC) [Pa] ;
- tensile strength, σ_t (FT) [Pa] ;
- Strength in biaxial compression, σ_{bc}(FBC) [Pa] ;
- stresses at failure during a triaxial test S1, S2, S3 (SIG1,SIG2,SIG3) [Pa] ;
- (relative) initial elastic limit A_o(A0) [-] ;
- (relative) final elastic limit B_o(B0) [-] ;
- hardening factor κ (KAPPA) [-].

3.2.23. Bultel (1999) swelling model (IMOD=98)

F. Bultel (1999) has developed an anistropic non linear elastic-perfectly plastic model that accounts for a volume strain associated with the decrease of the mean effective stress.

Parameters for the Bultel swelling model (IMOD=98) :

- pression de gonflement ;
- indice de gonflement ;
- anisotropie de gonflement ;

- nx, ny, nz

3.3. "User-defined" models

3.3.1. Principle

In order to make it possible to test the influence of the choices made to model the behaviour of materials (and also to simplify the introduction of new models), a new structure was developed for the constitutive models, in which the user is invited to define the constitutive model as the combination of an elasticity chosen from a predefined list (linear or not, isotropic or not), and of one or more plastic mechanisms (each defined by a criterion, a flow rule, and if necessary a hardening law).

Each of these elements is to be chosen from a list of pre-existing models, but the possibility of combining the elements of behavior makes it possible to use a wide range of models.

The models are available for bulk elements, in 2D and in 3D.

They can be used with MCNL, TCNL, CSNL modules. Heterogeneous linear elastic models can be used with DYNI.

In the present state, these models cannot be used with other mechanical modules (MEXO, MCCI, MPNL, FLAM, SUMO).

3.3.2. Implementation in the datafile

In practice, their use consists in describing in the data file, the constitutive model in the form of a succession of lines, each associated with one aspect of the law considered. The graphical interface allows you to use this combination of models.

In the data set, the nature of the information provided by each line is indicated by a keyword of 3 or 4 letters (ELAS for the elastic law, RHO for the density, etc.). This keyword is followed by an integer indicator and a variable number of numeric values. In other words, the IMOD indicator is split into a series of indicators related to the elastic part of the model, the criterion(s) and the plastic potential(s), to the presence of reinforcement inclusions, etc.

More precisely, the user defined models are associated with the value IMOD = 10000.

For a mechanical problem, the only mandatory information concerns the law of elasticity, necessary to calculate a stiffness matrix. The other informations are optional: there is no need to specify a plastic mechanism for an elastic material.

The recognized keywords are:

- RHO : density,
- ELAS : elastic model,
- CRT, POT, ECR : criterion, plastic potential, hardening law of the first plastic mechanism,
- CRT2, POT2, ECR2 : description of the second plastic mechanism (if required),
- RENF : models for the reinforcement inclusions, taken into account with the same technique as in the classical IMOD=43 model (cf. 3.2).
- NDR : indicator for undrained simulations.

The keywords are sought for in the order: RHO, ELAS, CRT, POT, ECR, CRT2, POT2, ECR2, NDR. The data provided by the user can omit one or several keywords, but not modify their order.

As an example, in plane strain, for a ground whose behaviour is described by the Mohr Coulomb model with ρ = 20 kN/m3 ; E = 100 MPa ; ν = 0,3 ; c= 150 kPa ; φ = ψ = 25 degrés, one finds in the data file (if the unit for stresses is MPa), in the case of the classical models of section 3.2 :

```
ground1                                         A          9
```

```
10      1       0.02    100.    0.3     0.15    25.     25.
```

For the same theoretical model and the same parameters, in the case of the user defined models, the data file comprises the following lines:

```
ground1                                     A           9
10000   1
RHO     0.02
ELAS    0       100.    0.3
CRT     4       0.15    25.
```

The line beginning with RHO introduces the value of the density.

The line beginning with ELAS describes the elastic part of the model. The indicator after ELAS is set to zero in the case of a linear isotropic elasticity, and is followed by the values of E and ν.

The line beginning with CRT indicates that there is a plastic mechanism and defines the chosen plasticity criterion. The indicator 4 corresponds to the Mohr Coulomb plasticity criterion. It is followed by the values of the parameters associated with this model, c and φ.

Since there are no additional lines, the flow rule is associated and the model is perfectly plastic (no hardening). To define a model in which the dilatancy angle ψ is different from the friction angle φ, the data would be supplemented by another ligne beginning with the keyword POT).

In a general way, for the criterion, the data are given under the following form

```
CRT ICRIT  parameters
```

If no plastic potential is explicitly introduced, the flow rule is associated. For a non associated flow rule, the user specifies a plastic potential by adding a line:

```
POT IPOT  parameters
```

In a similar way, if no hardening law is explicitly introduced, the model is perfectly-plastic. Othervwise, the parameter defining the elastic domaine may vary.

The typical hardening model is the modified Cam Clay model, the yield surface is an ellipse in the (p',q) plane. In isotropic compression, if the mean stress becomes large enough, irreversible volumetric strains occur and the elastic domain expands. The parameter M that defines the eccentricity of the ellipse does not vary, but the parameter p_c (the major axis) increases.

This can be modelled by an exponential relation between p_c and the plastic volumetric strain:

$$p_C = p_C{}^{\circ} \exp \left[- \frac{(1+e_o)\, \varepsilon_v{}^p}{(\lambda-\kappa)} \right]$$

This relation is called hardening law. Other formulations could involve a more complex relation between p_c and the volumetric strain, or modify the value of M. Different hardening laws can be associated to a given criterion. As regards the format of the data, the user adds another line

```
ECR   IECR  parameters
```

For the previous hardening law, the user inputs the values of the extra parameters λ, κ, e_o.

3.3.3. Elasticity models

CESAR proposes a set of elastic models, linear or not, isotropic or not, with homogeneous or heterogeneous moduli inside the group of elements.

In the case of non linear elastic models, the formulation generally gives the value of tangent moduli depending on the stres state (and/or on the strain state). Numerically, the elastic non-linearities are taken into account by updating the moduli at the beginning of each increment of computations, on the basis of the stresses and strains obtained at the end of the previous increment.

3.3.3.1. Linear isotropic elasticity (IELAS=0)

The formulation is identical to that of the « classical » model IMOD=1 (cf 0), but it can be combined with other features (such as one or two plastic mechanisms).

Formulation :

$$\varepsilon^e = \frac{1+\nu}{E}\,\sigma - \frac{\nu}{E}\,\mathrm{tr}(\sigma)\,1$$

> Parameters for the linear isotropic elastic model (IELAS=0):
>
> •Young's modulus E (`YOUNG`) [Pa]
>
> • Poisson's ratio ν (`POISS`) [-]

3.3.3.2. Linear isotropic elasticity with moduli varying with depth (IELAS=1)

It is generally accepted that elastic modules increase with depth, because of the gradual increase in vertical stress, and the densification that can result. In situ tests (for instance pressuremeter profiles) are generally in agreement with this hypothesis. CESAR proposes to take into account a linear variation of Young's modulus E and possibly of Poisson's ratio with depth.

Formulation :

$$E = E_o + \Delta E\ z \quad ; \qquad \nu = \nu_o + \Delta\nu\ z$$

where z denotes the coordinate of the considered point along the upward vertical direction.

> Parameters for the linear isotropic elasticity with moduli varying with depth (IELAS=1):
>
> •Young's modulus for z=0, E_o (`YG0`) [Pa]
>
> • Poisson's ratio for z=0, ν_o (`NU0`) [-]
>
> • gradient of Young's modulus along the upward vertical direction ΔE (`DELTAYG`) [Pa/m]
>
> • gradient of Poisson's ratio along the upward vertical direction $\Delta\nu$ (`DELTAP`) [m^{-1}]
>
> Caution : To account for the fact that the stiffness increases with depth (i.e. as z decreases), the vertical gradient must be **negative** : $\Delta E < 0$.

3.3.3.3. Linear isotropic elasticity with E in power of z (IELAS=8)

This model is a variant of the previous one, in which the variation of the modulus with the depth is not linear. The formulation adopts a constant modulus value E0 for the points located above an altitude h, and a value varying according to a power law below this altitude.

Formulation :

$$\nu = \text{constant} \qquad ; \qquad E = E_0 + k\,\langle h-z\rangle^{\alpha} \quad \text{where } \langle h-z\rangle = \frac{h-z+|h-z|}{2}$$

where z denotes the coordinate of the considered point along the upward vertical direction.

$\langle h-z \rangle$ is the positive part of h-z :

$$\langle h-z \rangle = h - z \text{ if } h > z \quad ; \quad \langle h-z \rangle = 0 \text{ if } h \le z$$

E is therefore equal to $E_0 + k\,(h-z)^\alpha$ if z is less than h, and E_0 if z greater than h.

Parameters for the Linear isotropic elasticity with E in power of z (IELAS=8):

- Young's modulus for z=h, E_o (YG0) [Pa]
- coefficient for the variations of Young's modulus k (K) [Pa. $m^{-\alpha}$]
- reference altitude h (H) [m]
- exponent α [-]
- Poisson's ratio ν [-]

NB : If the origin of the vertical coordinates is shifted, the value of h must be modified accordingly.

NB2 : For this model, a moduli increasing as depth increases (i.e. when z decreases), parameter k must take a **positive** value.

3.3.3.4. Linear transversely isotropic elasticity (IELAS=2)

In CESAR, the local axes attached to the material directions are denoted 1, 2, 3. The revolution symmetry axis is number 2, the behaviour is isotropic in the plane perpendicular to the unit vector e_2. In ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$), the stress-strain relation is given by:

$$\begin{bmatrix} \varepsilon_{33} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_1/E_1 & -\nu_2/E_2 & 0 & 0 & 0 \\ -\nu_1/E_1 & 1/E_1 & -\nu_2/E_2 & 0 & 0 & 0 \\ -\nu_2/E_2 & -\nu_2/E_2 & 1/E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu_1)/E_1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{33} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix}$$

The model is defined by 5 parameters E_1, E_2, ν_1, ν_2 and G, and by the orientation of the axis with respect the (e_x, e_y, e_z) in which the computation is carried out (characterized by one angle in plane strain and two angles in 3D).

Parameters for the linear transversely isotropic (IELAS=2)

- Young's modulus in the isotropic plane E_1 (E1) [Pa]
- Young's modulus in the direction of the symmetry axis E_2 (E2) [Pa]
- Poisson's ratios in the isotropic plane ν_1 (P1) [-] and transverse ν_2 (P2) [-]
- shear modulus G (G) [Pa]
- in bidimensional condition :

  angle θ between OX and the isotropic plane (TETA) [deg]
- in three-dimensional condition :

> angle θ between OX and the projection of the symmetry axis onto the horizontal plane (`TETA`) [deg]
>
> angle φ between the horizontal plane and the isotropic plane (`PHI`) [deg]

3.3.3.5. Heterogeneous transversely isotropic elasticity - chevron-shaped fractures (IELAS=9)

This model was proposed by A. Pouya to take into account the influence of fractures on the mechanical behaviour of argillites in the vicinity of an excavation. The digging of a gallery causes the creation of a zone called Excavation Damaged Zone or EDZ, traversed by fractures which have a complex three-dimensional geometry.

The IELAS models = 9/10 represent a simplified geometry which corresponds to fractures having a conical shape (symmetry of revolution around the axis of the gallery and constant inclination with respect to the axis). Moreover, the spacing between the fractures is also assumed to be constant, and equal to D.

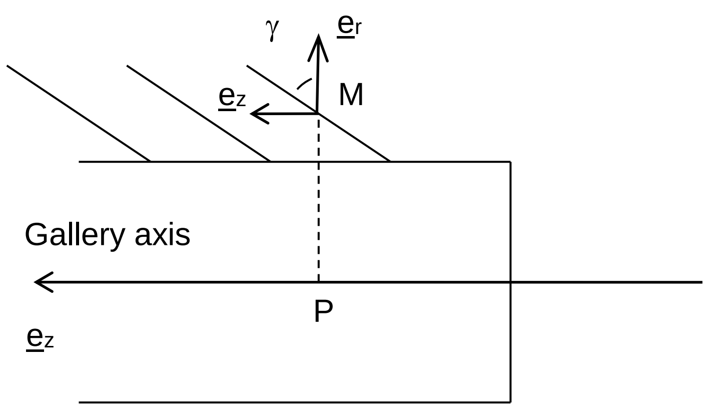


Figure 1 – Definition of the geometry of the chevron-shaped fractures

For fractures oriented as in the figure (with the top of the cones on the side of the gallery's excavation face), the angle γ takes a positive value. If we change the orientation of the gallery axis, the value of γ corresponding to the same geometry of fractures is negative.

The influence of the fractures on the mechanical behavior of the massif is taken into account in a homogenized manner, by attributing to the fractured soil an anisotropic equivalent behavior, in directions which depend on the position of the point considered.

The model is a transversely isotropic elastic model, with the particularity that the local direction of the axis of symmetry varies from one point to the other of the massif. It takes the following form:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}$$

Where the direction of the revolution symmetry axis is z, and where the coefficients can be derived from the elastic properties of the intact rock E and ν, and from the fractures charactestics through:

$$s_{11} = 1/E \; ; \; s_{12} = s_{13} = -\nu/E \; ; \; s_{33} = \frac{1}{E} + \frac{1}{k_n D} \; ; \; s_{44} = \frac{2(1+\nu)}{E} + \frac{1}{k_t D} \; ; \; s_{66} = 2\,(s_{11}-s_{12}) = \frac{2(1+\nu)}{E}$$

In the relations above, k_n and k_t denote respectively the normal and tangential stiffnesses of the fractures.

The proposed model has the same form as that of section 3.3.3.4 (IELAS=2), if direction 2 of IELAS=2 is identified with direction z of IELAS=9, if we let:

$$E_1 = E \; ; \; \nu_1 = \nu \; ; \; E_2 = \frac{1}{\frac{1}{E} + \frac{1}{k_n D}} \; ; \; \nu_2 = \nu\frac{E_2}{E} = \frac{\nu}{1 + \frac{E}{k_n D}} \; ; \; G = \frac{1}{\frac{2(1+\nu)}{E} + \frac{1}{k_t D}}$$

If IELAS is set to 9, the user inputs the 5 usual parameters (E_1, E_2, ν_1, ν_2 , G) calculated using the above formulae. If IELAS is set to 10, the user inputs the 5 parameters (E, ν, k_n, k_t , D), and the solver computes the value of the equivalent homogenized moduli.

In the last place, it is necssary to define the orientation of the isotropic plane (or of the revolution axis).

in 2D : γ is the angle between the radial unit vector er and the isotropic plane (tangent to the fractures)

in 3D : the user specifies the coordinates (x,y,z) of one point of the symmetry axis, and two angles α, β defining its direction (the unit vector of the axis si (cos α cos β, sin α cos β, sin β)), , and the angle γ between the radial and the isotropic plane.

Parameters for the transversely isotropic elasticity - chevron-shaped fractures model (IELAS=9)

- Young's modulus in the isotropic plane E_1 (`E1`) [Pa]
- Young's modulus in the direction of the symmetry axis E_2 (`E2`) [Pa]
- Poisson's ratios in the isotropic plane ν_1 (`P1`) [-] and transverse ν_2 (`P2`) [-]
- shear modulus G (`G`) [Pa]

In axisymmetric condition :

- angle between the radial unit vector er and the isotropic plane γ [deg]

In three dimensional condition:

- coordinates of one point of the gallery axis X,Y,Z [m]
- angles defining the axis orientation α,β [deg]
- angle between the radial unit vector er and the isotropic plane γ [deg]

Parameters for the homogenized transversely isotropic elasticity - chevron-shaped fractures model (IELAS=10)

- Young's modulus of the intact rock (`E`) [Pa]
- Poisson's ratios the intact rock ν [-]
- normal and tangential stiffnesses of the fractures k_n and k_t [Pa/m]
- distance between fractures D [m]

In axisymmetric condition :

- angle between the radial unit vector er and the isotropic plane γ [deg]

In three dimensional condition:

- coordinates of one point of the gallery axis X,Y,Z [m]
- angles defining the axis orientation α,β [deg]
- angle between the radial unit vector er and the isotropic plane γ [deg]

3.3.3.6. Transversely isotropic elasticity for curved fractures (IELAS=11)

p.m. : It is a variant of the previous model, in which the elastic matrix is tranversely isotropic, with a symmetry of revolution around the gallery axis, but the shape of the fractures is more complex.

3.3.3.7. Orthotropic linear elasticity (IELAS=14)

The orthotropic elastic behaviour corresponds to materials having three planes of symmetry. There exists a trihedron of orthonormal vectors ($\underline{e}_b$, $\underline{e}_h$, $\underline{e}_n$), which we can take with a direct orientation, such that the following holds (if we let $\varepsilon_{\alpha\beta} = \underline{e}_\alpha \cdot \varepsilon \cdot \underline{e}_\beta$ and $\sigma_{\alpha\beta} = \underline{e}_\alpha \cdot \sigma \cdot \underline{e}_\beta$) :

$$\begin{bmatrix} \varepsilon_{bb} \\ \varepsilon_{hh} \\ \varepsilon_{nn} \\ 2\varepsilon_{bh} \\ 2\varepsilon_{hn} \\ 2\varepsilon_{nb} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_b} & -\frac{\nu_{hb}}{E_h} & -\frac{\nu_{nb}}{E_n} & 0 & 0 & 0 \\ -\frac{\nu_{bh}}{E_b} & \frac{1}{E_h} & -\frac{\nu_{nh}}{E_n} & 0 & 0 & 0 \\ -\frac{\nu_{bn}}{E_b} & -\frac{\nu_{hn}}{E_h} & \frac{1}{E_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{bh}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{hn}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{bn}} \end{bmatrix} \begin{bmatrix} \sigma_{bb} \\ \sigma_{hh} \\ \sigma_{nn} \\ \sigma_{bh} \\ \sigma_{hn} \\ \sigma_{nb} \end{bmatrix}$$

Given the symmetries of the elastic tensors (Salençon, 2007) :

$$\nu_{hb} E_b = \nu_{bh} E_h \quad ; \quad \nu_{nh} E_h = \nu_{hn} E_n \quad ; \nu_{nb} E_b = \nu_{bn} E_n$$

There is no symmetry of the Poisson's ratios ν_{ij} and ν_{ji} and special attention must be paid to the indices.

On the other hand, this relation makes it possible to interpret relatively simply the various coefficients: if one imposes a state of uniform stress defined by $\sigma_{\alpha\alpha} = \Sigma$ in one of the directions (α = b, h or n), we obtain a strain in the same direction equal to $\varepsilon_{\alpha\alpha} = \Sigma / E_\alpha$ and in the transverse direction γ a strain equal to $\varepsilon_{\gamma\gamma} = -\nu_{\gamma\alpha}\Sigma / E_\gamma$.

In practice, in the numerical procedures, the opposite relation is used (Assire and al, 2010):

$$\begin{bmatrix} \sigma_{bb} \\ \sigma_{hh} \\ \sigma_{nn} \\ \sigma_{bh} \\ \sigma_{hn} \\ \sigma_{nb} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{1-\nu_{hn}\nu_{nh}}{E_h E_n} & \frac{\nu_{hb}+\nu_{nb}\nu_{hn}}{E_h E_n} & \frac{\nu_{nb}+\nu_{hb}\nu_{nh}}{E_h E_n} & 0 & 0 & 0 \\ \frac{\nu_{bh}+\nu_{bn}\nu_{nh}}{E_b E_n} & \frac{1-\nu_{nb}\nu_{bn}}{E_b E_n} & \frac{\nu_{nh}+\nu_{nb}\nu_{bh}}{E_b E_n} & 0 & 0 & 0 \\ \frac{\nu_{bn}+\nu_{bh}\nu_{hn}}{E_b E_h} & \frac{\nu_{hn}+\nu_{hb}\nu_{bn}}{E_b E_h} & \frac{1-\nu_{bh}\nu_{hb}}{E_b E_h} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{bh}\Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{hn}\Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{bn}\Delta \end{bmatrix} \begin{bmatrix} \varepsilon_{bb} \\ \varepsilon_{hh} \\ \varepsilon_{nn} \\ 2\varepsilon_{bh} \\ 2\varepsilon_{hn} \\ 2\varepsilon_{nb} \end{bmatrix}$$

where $\Delta = \dfrac{1 - \nu_{hn}\nu_{nh} - \nu_{nb}\nu_{bn} - \nu_{bh}\nu_{hb} - 2\,\nu_{hn}\nu_{nb}\nu_{bh}}{E_b\, E_h\, E_n}$

Note: in the particular case of the plane strains (in the $(\underline{e}_b, \underline{e}_h)$ plane), one has:

$$\begin{bmatrix} \sigma_{bb} \\ \sigma_{hh} \\ \sigma_{bh} \\ \sigma_{nn} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{1-\nu_{hn}\nu_{nh}}{E_h\, E_n} & \frac{\nu_{hb}+\nu_{nb}\nu_{hn}}{E_h\, E_n} & 0 \\ \frac{\nu_{bh}+\nu_{bn}\nu_{nh}}{E_b\, E_n} & \frac{1-\nu_{nb}\nu_{bn}}{E_b\, E_n} & 0 \\ 0 & 0 & G_{bh}\,\Delta \\ \frac{\nu_{bn}+\nu_{bh}\nu_{hn}}{E_b\, E_h} & \frac{\nu_{hn}+\nu_{hb}\nu_{bn}}{E_b\, E_h} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{bb} \\ \varepsilon_{hh} \\ 2\varepsilon_{bh} \end{bmatrix}$$

Note 2 : For the quadratic form defining the elastic energy to be defined and positive, the following condition must be fulfilled:

$1 - \nu_{bn}\nu_{nh}\nu_{hb} - \nu_{bh}\nu_{hn}\nu_{nb} - \nu_{bh}\nu_{hb} - \nu_{hn}\nu_{nh} - \nu_{nb}\nu_{bn} > 0$

Or more simply $\Delta > 0$

In the three-dimensional case, the model is defined by the three moduli E_b, E_h, E_n, the three Poisson's ratios ν_{bh}, ν_{hn}, ν_{bn}, the three shear moduli G_{bh}, G_{hn}, G_{bn}, and three angles defining the orientation of the local axes $(\underline{e}_b, \underline{e}_h, \underline{e}_n)$ in the global axes. The angles adopted here, denoted by ψ, θ, φ, are Euler's angles (the definition is recalled in appendix). This choice is made because it was adopted in other parts of the code for anisotropic plastic.

On the whole, the model depend on 12 parameters: 9 are mechanical parameters and 3 are geometric parameters.

In the bidimensional case, one one of the shear moduli is required, and the orientation of the local axis is also simplified. direction n is assumed to coincide with the plane in which the mesh is defined; since ($\underline{e}_b$, $\underline{e}_h$, $\underline{e}_n$) is direct, one only has to define the orientation of $\underline{e}_b$ in this plane. The user provides the angle θ between the horizontal axis Ox and $\underline{e}_b$

As a conclusion, the required parameters are:

in 2D : E_b, E_h, E_n, ν_{bh}, ν_{hn}, ν_{bn}, G_{bh}, θ

in 3D : E_b, E_h, E_n, ν_{bh}, ν_{hn}, ν_{bn}, G_{bh}, G_{hn}, G_{bn},ψ, θ, φ

Parameters for the general orthotropic elastic model (IELAS=14) in bidimensional condition

- Young's moduli in directions b, h,n : E_b, E_h, E_n (`EB,EH,EN`) [Pa]
- Poisson's ratios ν_{bh},ν_{hn},ν_{bn} [-]
- shear modulus in the bh plane G_{bh} [Pa]
- angle between Ox and direction b : θ [deg]

Parameters for the general orthotropic elastic model (IELAS=14) in three-dimensional condition

- Young's moduli in directions b, h,n : E_b, E_h, E_n (`EB,EH,EN`) [Pa]
- Poisson's ratios ν_{bh},ν_{hn},ν_{bn} [-]
- shear moduli G_{bh},G_{hn},G_{bn} [Pa]
- Euler's angles defining the orientation of the local axes (eb,eh,en) : ψ, θ, φ [deg]

3.3.3.8. Zucchini model for masonry (IELAS=15)

Zucchini and Lourenço (2002) proposed a model to take into account the anisotropy of the mechanical properties of masonry structures. It is an orthotropic model, like the previous one, whose Young's moduli, Poisson's ratios and shear moduli are deduced from the geometry of the brick arrangement (Figure 9) and from the elastic properties of the blocks and joints by a homogenization approach (NB: the model relies on an approximate resolution of the equations posed on the reference cell, but which makes it possible to obtain the desired characteristics in a efficient way from a numerical point of view).

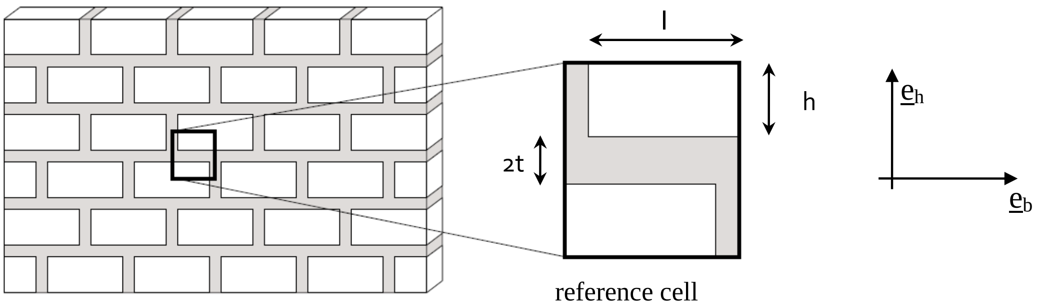


Figure 9 – Definition of the reference cell by Zucchini and Lourenço (2002)

In the terminology of masonry structures, the horizontal joints are called “bed joints”, the vertical joints are called "head joints", which justifies the indices b and h associated in what precedes the two first directions of the local axes. The last index, n, corresponds to the direction of the normal to the plane of the masonry wall.

The parameters to be provided are the mechanical characteristics (Young's modulus and Poisson's ratio) of the blocks (or bricks) and of the mortar which constitutes the joints:

in 2D : E_1, ν_1, E_m, ν_m, 2l, 2h, 2t, θ

in 3D : E_1, ν_1, E_m, ν_m, 2l, 2h, 2t,ψ, θ, φ

where index 1 is associated with blocks (or bricks) and index m with mortar joints.

The angles defining the orientation of the local axes attached to the masonry structure, denoted by ψ, θ, φ are Euler’s angles (whose definition is recalled in appendix).

Parameters for Zucchini model for masonry (IELAS=15)

- Young’s modulus of the bricks E_1[Pa]
- Poisson’s ratio of the bricks ν_1[-]
- Young’s modulus of the mortar E_m [Pa]
- Poisson’s ratio of the mortar ν_m [-]
- length, thickness of the bricks 2l, 2h [m]
- thickness of the joints 2t [m]

In bidimensional condition:

- angle between Ox and direction b : θ [deg]

In three-dimensional condition:

- Euler’s angles defining the orientation of the local axes (eb,eh,en) : ψ, θ, φ [deg]

3.3.3.9. Linear orthotropic elasticity for an elliptic vault (IELAS=16)

This model aims to reproduce the behaviour of a heterogeneous orthotropic material, whose local anisotropy directions follow an elliptical arch. The idea is to represent a masonry vault, the bricks of which would be arranged along the tangent and normal to an "average ellipse" which corresponds to the center line of the vault. The first local direction of anisotropy is parallel to the tangent to the ellipse, the second to the normal to the ellipse, the third to the axis of the gallery. The user is led to give a certain number of geometrical parameters, represented, in 2D, on the following figure.

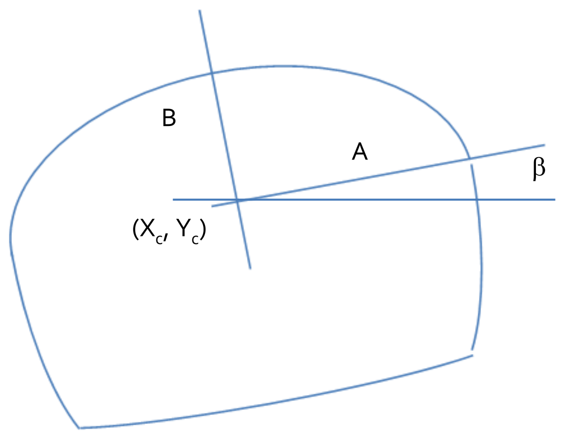


Figure 10 – Geometric parameters of the elliptical vault

In three dimensional condition, we give the coordinates (Xc, Yc, Zc) of a point on the gallery axis, two angles γ and ω (in degrees) which define the orientation of the gallery axis. γ is the angle (in the interval [-90 °, 90 °]) between the projection of the axis on the plane z = 0 and the x-axis (longitude). ω is the angle (in the interval [-90 °, 90 °]) between the direction of the axis and the horizontal (latitude). The components of a unit vector of the axis direction are therefore given by (cos γcosω, sin γ cosω, sin ω).

Parameters for the orthotropic elastic model for an elliptical vault (IELAS=16) in bidimensional condition

- Young's moduli in directions 1, 2, 3 : E_1, E_2, E_3 (`E1,E2,E3`) [Pa]
- Poisson's ratios ν_{12},ν_{23},ν_{13} [-]
- shear modulus in the plane 12 G_{12} [Pa]
- coordinates of the gallery axis Xc, Yc [m]
- major semi-axis and minor semi-axis of the ellipse A, B [m]
- angle between Ox and the major axis : β[deg]

Parameters for the orthotropic elastic model for an elliptical vault (IELAS=16) in three-dimensional condition

- Young's moduli in directions 1, 2, 3 : E_1, E_2, E_3 (`E1,E2,E3`) [Pa]
- Poisson's ratios ν_{12},ν_{23},ν_{13} [-]
- shear moduli G_{12} , G_{23} , G_{13} [Pa]
- coordinates of one point of the gallery axis Xc, Yc, Zc [m]
- major semi-axis and minor semi-axis of the ellipse A, B [m]
- angles defining the direction of the gallery axis γ, ω [deg]
- tilt angle: β[deg]

3.3.3.10. Zucchini model for an elliptical masonry vault (IELAS=17)

It is a variant of the previous model, in which the elastic moduli are derived from the characteristics of the masonry using the formulae proposed by Zucchini and Lourenço.

Parameters for the Zucchini model for an elliptical masonry vault (IELAS=17) in bidimensional condition

- Young's modulus of the bricks E_B[Pa]
- Poisson's ratio of the bricks ν_B [-]
- Young's modulus of the mortar E_m[Pa]
- Poisson's ratio of the mortar ν_m [-]
- length, thickness of the bricks 2l, 2h [m]
- thickness of the joints 2t [m]
- coordinates of the gallery axis Xc, Yc [m]
- major semi-axis and minor semi-axis of the ellipse A, B [m]
- angle between Ox and the major axis : β[deg]

Parameters for the Zucchini model for an elliptical masonry vault (IELAS=17) in thee-dimensional condition

- Young's modulus of the bricks E_B[Pa]
- Poisson's ratio of the bricks ν_B[-]
- Young's modulus of the mortar E_m[Pa]
- Poisson's ratio of the mortar ν_m [-]
- length, thickness of the bricks 2l, 2h [m]
- thickness of the joints 2t [m]
- coordinates of one point of the gallery axis Xc, Yc, Zc [m]
- major semi-axis and minor semi-axis of the ellipse A, B [m]
- angles defining the direction of the gallery axis γ, ω [deg]
- tilt angle: β[deg]

3.3.3.11. Non-linear « Cam-Clay type » isotropic elasticity (IELAS=3)

The previous elastic models are all linear, the distinction between them coming from the isotropic character or not, and from the possibility of taking into account heterogeneous characteristics in the same group of elements.

In the continuation, non-linear isotropic elastic models are presented, in which the elasticity moduli depend on the stress state.

In the case of the IELAS = 3 model, the compression modulus depends (linearly) on the average stress, and the shear modulus is constant.

The tangent bulk modulus (or compression modulus) K_t is propotional to the mean stress:

$$K_t = p\,(1+e_o)/\kappa$$

where e_o denotes the void ratio and κ a dimensionless parameter (sometimes called swelling coefficient).

The implementation in CESAR includes a « safety » to avoid negative or too small moduli: K_t is bound to remain larger than a prescribed value K_{min} .

The shear modulus G is constant. The (tangent) values of Young's modulus and Poisson's ratio are obtained using the classical relations:

$$E = 9GK/(3K+G) \text{ and } \nu = (3K-2G)/2(3K+G)\,).$$

> Parameters for the Non-linear « Cam-Clay type » isotropic elasticity (IELAS=3):
>
> - shear modulus G (G) [Pa]
> - void ratio e_o (EZERO) [-]
> - slope of the elastic unloading/reloading curves in isotropic compression κ (KAPPA) [-]
> - minimal value for the compression modulus K_{min} (KMIN) [Pa]
>
> NB : since K varies and G is constant, Poisson's ratio varies. As long are K and G are positive, the elastic matrix is definite and positive, but Poisson's ratio may become negative. This model must therefore be used with caution.

3.3.3.12. Non linear « Fahey and Carter type » isotropic elasticity (IELAS=4)

This model is a nonlinear isotropic elastic model, in which the tangent shear and compression moduli depend on the current stresses. The model is derived from Duncan's hyperbolic model. The shear modulus and compression vary as a power of the mean stress. The dependence on the deviatoric stress is more complex. It has a large number of parameters which give it flexibility. In return, several sets of parameters can give similar results for the simulation of a triaxial test by example, which makes the determination of the parameters from test results a difficult task. The results obtained with this model, as implemented in CESAR, are discussed in detail in the PhD thesis by Coquillay (2005).

The parameters are denoted by n, ν_o, f, g, C, p_{ref}, c', φ'.

The last two are the parameters of shear strength of the Mohr-Coulomb criterion (cohesion and friction angle), which are considered here as parameters also having an influence on the elastic properties. p_{ref} is a reference pressure value; n an exponent controlling the variations of G with the mean stress; C a reference shear modulus. The two scalars f and g are dimensionless.

Formulation :

$$G_o = C\,(1+\langle p\rangle/p_{ref})^n \qquad \text{where} \quad \langle p \rangle = p \text{ si } p \geq 0 \text{ and } 0 \text{ si } p < 0$$

$$G_t = G_o[(1-f\,(t/t_{max})^g]^2 \,/\, [(1-f(1-g)\,(t/t_{max})^2]$$

$$K_t = K_o = \frac{2(1+\nu_o)}{3(1-2\nu_o)}\,G_o$$

$$t = (\sigma_1-\sigma_3)/2 \text{ and } t_{max} = \frac{3(p'\sin\varphi' + c'\cos\varphi')}{3-\sin\varphi'}$$

Parameters for the Non linear isotropic « Fahey and Carter type » elasticity (IELAS=4):

- exponent n (N) [-]
- reference Poisson's ratio ν_o(NUZERO) [-]
- shape parameters f and g [-]
- reference shear modulus C [Pa]
- reference pressure p_{ref} (PREF) [Pa]
- cohesion c' (COHES) [Pa]
- friction angle φ' (PHI) [Pa]

3.3.3.13. Non linear elasticity of the Hardening soil model (IELAS=5)

CESAR makes it possible to consider the elastic part of the Hardening Soil Model (Schanz and al, 1999). It is a non linear elastic model, with a constant value of Poisson's ratio and a Young's modulus depending on the stress state. Like in other models (for instance the Fahey and Carter model), it also depends on strength properties c and φ. The value of Young's modulus is denoted by E_{ur} and given by :

$$E_{ur} = E_{ur}^{ref} \left(\frac{\sigma_3 + c \cot an\ \varphi}{\sigma_{ref} + c \cot an\ \varphi} \right)^m$$

Poisson's ratio is constant and denoted ν_{ur}.

The dependency of the modulus with the minor principal stress may give a zero or indetermined value. In CESAR, a minimal value is considered, equal to $E_{urref}/500$.

Note : the indices « ur » refer to an elastic unloading-reloading process.

Parameters for the Non linear elasticity of the Hardening soil model (IELAS=5):

- reference Young's modulus Eur_ref [Pa] ;
- Poisson's ratio in unloading-reloading ν_{ur} [-] ;
- cohesion c [Pa] ;
- friction angle φ [deg] ;
- exponent m [-] ;
- reference pressure p_{ref} [Pa]

3.3.3.14. Non linear « Duncan type » isotropic elasticity (IELAS=6)

In this model, the shear modulus decreases as one gets close to failure (describes by the Hoek and Brown model). The formulation is relatively complex :

- The tangent bulk modulus K_t is given by :

  $K_t = C\ (p/p_{ref})^n$ (with the additional constraint that $K_t \geq K_{min}$)

- The tangent shear modulus G_t is given by :

  $G_t = G_o\ (1\text{-}R_f\ (\sigma_1\text{-}\sigma_3)/(\ (\sigma_1\text{-}\sigma_3)_{rup}\ )$

  with $G_o = K_t\ (1\text{-}2\nu_o)/(2(1+\nu_o))$ and $(\sigma_1\text{-}\sigma_3)_{rup} = \sigma_c\ (m\ \sigma_3\ /\sigma_c + s)\text{^}\alpha$

Parameters for the Non linear « Duncan type » isotropic elasticity (IELAS=6):

- strength parameter σ_c (`SIGMAC`) [Pa] ;
- fracture coefficient s (`S`) [-] ;
- shape parameter m (`M`) [-] ;
- exponent of the Hoek Brown model α [-] ;
- minimal value of the bulk modulus K_{min} [Pa] ;
- reference value of the bulk modulus C [Pa] ;
- reference pressure p_{ref} [Pa] ;
- exponent in the bulk modulus formula n [-] ;
- Poisson's ratio under isotropic stress [-] ;
- parameter for the shear modulus R_f ($0 < R_f > 1$) [-]

3.3.3.15. Modified « Fahey and Carter type » isotropic elasticity (IELAS=7)

This is a simplified version of the Fahey and Carter model (IELAS=4).

The tangent shear modulus G_t is given by :

$$G_t = G \exp(-q/\alpha)$$

where q is the deviatoric stress: $q = (3/2\, s_{ij} s_{ij})^{1/2}$

G depends on the mean stress through:

$$G = \text{Max}\{ G_{min},\ G_o (p/p_a)^n \}$$

Poisson's ratio ν is constant.

Parameters : G_o, p_a, n, G_{min}, α, ν

> Parameters for the modified Fahey and Carter isotropic elastic model (IELAS=7):
>
> • reference shear modulus G_o [Pa]
>
> • reference pressure p_a (PREF) [Pa]
>
> • exponent n (N) [-]
>
> • minimal value of G , G_{min} (GMIN) [Pa]
>
> • coefficient for the decrease of the shear modulus with q, α (ALPHA) [Pa]
>
> • Poisson's ratio ν [-]

3.3.3.16. Non linear transversely isotropic elasticity : ANL (or Gilleron) model (IELAS=24)

The most widely used model for the calculation of geotechnical structures is the elastic-perfectly plastic model obtained by combining an isotropic linear elastic law with the Mohr-Coulomb plasticity criterion, with a non-associated flow rule. This model does not necessarily give very good results to represent the distribution of settlements on the surface above a shallow tunnel, and tends to give excessively broad "settlement troughs": the area concerned by significant settlements on either side the tunnel is greater in the calculations than that shown by the available observations.

Gilleron (2017) proposed to use an anisotropic elasticity, without making the formulation too complex. The proposed model is an elastoplastic-perfectly plastic model whose elastic part is non-linear: the stiffness of the ground increases when the mean stress increases and decreases when the deviatoric stress increases. The originality of the formulation resides in the fact that it makes it possible to control the initial stiffness in the ground, under the assumption that the initial stresses are geostatic.

The model has several components:

Dependency of the stiffness with the mean stress

The shear modulus at small strain, denoted by G_0, is given by :

$$G_o = G_{ref} + A\,(p'/p_{ref})^r$$

Where p' is the mean stress, p_{ref} is a parameter for normalization, fixed at 20 kPa (roughly the vertical variation of the vertical stress over 1 m above the water table); A is a parameter homogeneous to a stress which controls the increase in shear modulus; r is a dimensionless exponent, which we propose to take r equal to 0.5 for granular soils and 1 for highly cohesive soils; G_{ref} is a reference value for the shear modulus, which corresponds to a zero mean stress. This is the initial shear modulus if we choose A equal to zero to deactivate the dependency of G_o on p'. This model introduces a stiffness gradient with depth which improves the estimation of settlements in the calculation of tunnels, by reducing settlements far from the axis. This helps to reduce the width of the settlement trough, but is not enough to get realistic widths.

Dependency of the moduli on the deviatoric stress

The curve (q - ε_1) takes the form of a curve that can be approached by a hyperbola whose asymptote is defined by the maximum admissible deviatoric stress, itself fixed by the plasticity criterion. In the ANL model, we normalize the deviatoric stress by its initial value or a value defined by the user, and we define an exponential decrease in the shear modulus as a function of the normalized deviatoric stress. This formulation enhances the strains near the excavation, where the ground is disturbed.

The tangent shear modulus is calculated according to:

$$G_t = Min\left[Max\left(G_{min}; G_0.\xi^{\left(1-\frac{q}{\beta_i p_i'}\right)}\right); G_0\right]$$

The Min and Max operators make it possible to ensure that G_t remains greater than a minimum value G_{min}, and less than a maximum value G_0. G_0 is defined from the initial stress state and the dependency formulation with the mean stress seen in the previous paragraph. G_{min} is the initial minimal shear modulus tangent. It is recommended to take G_{min}/G_o equal to 0.1: by limiting the shear modulus to 10% of its initial value, one avoids excessive elastic deformations. The deviatoric stress q is normalized by the product of the parameter β_i by the initial effective mean stress p'_i (and not by a value set by the user because this solution would not allow to take into account the vertical gradient of deviatoric stress).

The normalization of the deviatoric stress q involves the initial effective mean stress p'_i, which is zero at the surface. This makes it difficult to calculate the shear modulus when approaching the surface. We could adapt this formulation slightly (for example by taking G_t is equal to G_0 if p'_i 'is less than a certain value, or use a linear elastic law on the first two meters of the model where the ground is often altered and difficult to mechanically characterize with precision.

The parameter β_i describes the decrease in shear modulus as q increases. It is taken equal to:

$$\beta_i = Max(\frac{q_i}{p_i'}; \beta)$$

where β is the value of the ratio q/p'_i for which the decrease of the shear modulus decreases. β is defined by the user. By taking $\beta > q_i / p'_i$ it is possible to delay the reduction of the shear modulus, when the deviatoric stress has greatly increased. We can relate the parameter β to the friction angle by Jaky's formula, or to K_0^{NC}.

$$\beta = \frac{(1 - K_0^{NC})}{(1 + 2K_0^{NC})/3} = \frac{3 - 3(1 - sin\varphi)}{1 + 2(1 - sin\varphi)}$$

In the last place, the parameter ξ is a dimensionless coefficient which controls the decrease of the tangent shear modulus, if it is greater than 1. If ξ is less than or equal to 1, we come back to the linear elastic case with G_0 as shear modulus whatever the value of the other parameters: the mechanism for decreasing the stiffness with the deviatoric strain is deactivated.

This formulation makes it possible to master the initial module in the massif whatever the initial state of stress thanks to the parameter β and to isolate the contribution of the deviatoric stress in the degradation of this module. The parameter ξ allows to define the decay of the module independently of the failure parameters.

Poisson's ratio

Following the approach proposed by (Fahey & Carter, 1993), the tangent Poisson's ratio is variable, in such a way that the bulk modulus K_t remains constant :

$$\nu_t = \left[\frac{(1 + \nu_0) - (1 - 2\nu_0)G_t / G_0}{(2(1 + \nu_0) + (1 - 2\nu_0)G_t / G_0}\right]$$

Adopting a variable value of Poisson's ratio avoids a decrease in bulk modulus as the shear modulus decreases.

Extension : unloading/reloading feature

One can associate with this nonlinear elastic formulation an elastic module of unloading / reloading for the situations where the deviatoric stress is lower than its greatest known value. For this, an additional state variable q_e is introduced, which represents the greatest deviatoric stress undergone at the point considered. We introduce the modulus of elasticity of unloading / reloading G_{ur} as well as the Poisson's ratio of unloading / reloading ν_{ur}. This mechanism is activated when the deviatoric constraint q is less than 90% of q_e :

If $q \geq 0.9\ q_e$: primary deviatoric loading, one uses the preceding equations

if $q < 0.9.\ q_e$: unloading / reloading, the behavior is linear elastic with $G_t = G_{ur}$; $\nu_t = \nu_{ur}$

The state variable q_e is initialized equal to $\beta\ p'_i$, the largest past value of the deviatoric stress. It is updated at each calculation increment. Figure 2 illustrates the principle of this part of the model, by representing a curve (q, ε_1) during a non-monotonic triaxial test. Until reaching the initial deviatoric stress, the soil is in initial loading. Then the degradation of stiffness as a function of the deviatoric stress is activated according to the exponential law.

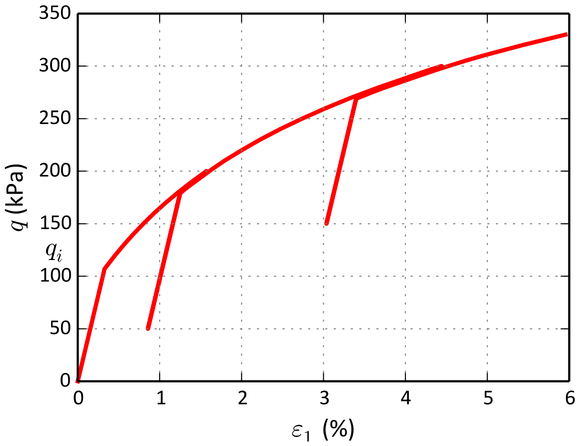


Figure2. Example of (q,ε_1) curve during a non monotonous triaxial test

Parametric studies carried out by (Gilleron, 2016) show that the impact of this mechanism, for a tunnel calculation, is negligible. Indeed, the displacements at the surface are generated ahead of the tunnel face and at the level of the unsupported zone, where the deviatoric stress increases monotonously with the increments of loadings and the advance in the phases of excavation.

Anisotropy of elastic properties

The point of view of Gilleron (2016) is that more realistic settlement troughs can be obtained if a transversely isotropic elasticity is taken into account in the calculation. The stress-strain relationship is given by:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix} = \begin{bmatrix} 1/E_h & -\nu_{hh}/E_h & -\nu_{vh}/E_v & & & \\ -\nu_{hh}/E_h & 1/E_h & -\nu_{vh}/E_v & & 0 & \\ -\nu_{vh}/E_v & -\nu_{vh}/E_v & 1/E_v & & & \\ & & & (1+\nu_{hh})/E_h & 0 & 0 \\ & 0 & & 0 & 1/2G_{vh} & 0 \\ & & & 0 & 0 & 1/2G_{vh} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

The behaviour is described by five independent parameters : E_h, E_v, ν_{hh}, ν_{vh} and G_{vh}.

In the literature, several ratios have been considered:

- the ratio between the horizontal and vertical Young's moduli $n = E_h/E_v$.

- the ratio between the horizontal and vertical shezar moduli $\alpha = \frac{G_{hh}}{G_{hv}}$, where

$$G_{hh} = \frac{E_h}{2(1+\nu_{hh})}$$

- the ratio of the vertical shear modulus to the vertical Young's modulus $m = G_{vh}/E_v$. For an isotropic model, it is equal to $m_{iso} = \frac{1}{2(1+\nu_{hh})}$, which gives 0.33 for an undrained ground(ν= 0,5) and 0,5 if ν is zero.

There is the following relation between the ratios defined above:

$$\frac{m}{m_{iso}} = \frac{n}{\alpha}$$

The influence of the anisotropy of elastic behavior on calculated settlements is not widely accepted in the literature, probably due to the difficulty of choosing the parameters. However, two approaches can be identified: the first focuses on the transverse shear modulus and assumes that the vertical and horizontal elastic moduli are equal, which may seem unrealistic; the second, based on the work of (Graham & Houlsby, 1983), establishes correlations between the five parameters. It leads to a vertical shear modulus superior to the shear modulus in the isotropic plane.

Gilleron proposes a formulation which makes it possible to reproduce the observed settlement troughs, when the pressure coefficient of the earth at rest is less than 1, which seems representative of the Paris region. We assume that this anisotropy of the stress state results in a horizontal Young's modulus weaker than the vertical Young's modulus, that is to say n <1. We limit a priori the interval of variation of n to values between 0.5 and 1. We then seek to define a correlation between n and m / m_{iso} which would involve α in the manner of the works of (Graham & Houlsby, 1983). We propose the following equality:

$$\alpha = \frac{1}{n^x}$$

This formula constitutes the basic hypothesis of the proposed model. It is justified by the good results obtained for the width of the settlement troughs, but is not supported by sufficient and documented adequate test results. Knowledge of two parameters, α and x by example then makes it possible to completely determine the transverse isotropic model. We have :

$$n = \alpha^{-1/x}$$

$$\frac{m}{m_{iso}} = \alpha^{-(1/x+1)}$$

If x is infinite (for given α), n tends towards 1 and m / m_{iso} tends towards 1 / α, we then find the model of (Lee & Rowe, 1989) where only the transverse shear modulus differs from the isotropic case. On the other hand, we can verify that n = E_h / E_v is an increasing function of x if α> 1 and decreasing if α <1.

By varying the value of α, for x fixed, we simultaneously vary n and m / m_{iso}. For example, for x = 0.8, when n varies between 0.5 and 0.8, the m / m_{iso} ratio varies between 0.2 and 0.6.

A parametric study conducted by Gilleron (2016) makes it possible to choose the n and m / m_{iso} parameters of the model to adjust the width of the Peck trough given in (Mair & Taylor, 1997) for different types of soil. This correlation was established, on the basis of numerical simulations, for a 10 m diameter tunnel whose axis is 20 m deep, with a linear elastic behavior model. It is therefore only indicative and its field of validity remains to be specified.

Table 1. Correlation between transverse isotropic parameters and the width parameter K of the Peck model

Nature of the ground	K	n	m/m_{iso}
Clays	0,5	1	1
Marls/silts	0,4	0,76	0,61
Medium sands	0,35	0,60	0,40
Rough sands	0,325	0,50	0,28

Non linear transversely isotropic elasticity : ANL (or Gilleron) model (IELAS=24):

- Gref [Pa] : reference value for the shear modulus
- A [-] : parameter for the increase in shear modulus with the mean stress
- r [-] : exponent for the increase in shear modulus with the mean stress
- Gmin/G0 [-]
- XSI [-] : shape factor for the decrease in shear modulus with the deviatoric stress
- BETA [-] : parameter for the decrease in shear modulus with the deviatoric stress
- nu0 [-] : reference Poisson's ratio
- Gur/G0 [-] : parameter for the unloading/reloading mechanism
- nu_ur : Poisson's ratio during unloading/reloading
- n : ratio of the horizontal modulus to the vertical modulus
- m/miso : parameter for the vertical shear modulus
- THETA, PHI [deg]

in bidimensional condition, θ is the angle between Ox and the isotropic plane

in three-dimensional condition, θ is the angle between the first axis of the global frame and the projection of the symmetry axis on the horizontal plane, and φ the angle between the horizontal plane and the isotropic plane

3.3.4. Plasticity criteria, plastic potentials, hardening laws

3.3.4.1. Tresca criterion (ICRIT=1)

The criterion is defined by:

$f(\underline{\sigma}) = \sigma_1 - \sigma_3 - 2\,c$

where σ_1 and σ_3 denote the largest and the small principal stress (in algebraic values).

The criterion limits the difference between the extreme principal stresses $\sigma_1 - \sigma_3$ (assuming that $\sigma_1 \geq \sigma_2 \geq \sigma_3$). The condition $f(\underline{\sigma}) \leq 0$ enforces that this difference remains less than 2c, where c is material property. It is readily seen that the maximum tensile stress in simple uniaxial tension is 2c, and Mohr representation of stresses shows that the maximum shear stress is equal to c (the diameter of the largest Mohr circle is equal to $\sigma_1 - \sigma_3$).

It can be noted that the value of the criterion is insensitive to the mean stress, i.e., for any value of p:

$f(\underline{\sigma}) = f(\underline{\sigma} + p\,\underline{1})$

This plasticity criterion is well adapted for quasi-incompressible materials, such as clays in undrained conditions, which led to call c cohesion in CESAR. It can also be used for metals (c would rather be called strength in pure shear).

> Parameters for the Tresca criterion (ICRIT=1):
>
> • cohesion c (C) [Pa]

Hardening laws for the Tresca criterion

No hardening law has been implemented for this criterion so far.

3.3.4.2. Critère de von Mises (ICRIT=2)

• Formulation: $f(\underline{\sigma}) = \sqrt{J_2} - k$ with $J_2 = 1/2\ \underline{s} : \underline{s}$

• Input parameters: k

This criterion is a condition on the second invariant of the stress tensor J2. In the context of the triaxial test, this invariant is proportional to the difference between the major and minor principal stresses. As for the Tresca criterion, the value of the yield function does not depend on the mean stress. The load surface is however different from that of the Tresca criterion, since it is a cylinder in the principal stresses space, whose axis coincides with the trissector.

The maximum stress in simple traction is equal to $k\sqrt{3}$, and the maximum shear stress is equal to k.

This criterion is rather used for metals.

Unlike the previous one, this criterion can be associated in CESAR with several hardening laws.

Hardening laws for the von Mises

• linear hardening: IECR = 1

It is the same hardening law as in the classical model IMOD=12 (see Mestat, 1993)

$$k + \Delta k = \sqrt{k^2 + 2\,A\,\Delta W^p}$$

where ΔW^p is the increment of plastic dissipation defined by: $\Delta W^p = \sigma : \Delta\varepsilon^p$

input parameter : A

• Prevost and Hoeg hardening law: IECR = 2

It is the same hardening law as in the classical model IMOD=19

$$k = \frac{A\,[B\,(\varepsilon_d{}^p)^2 + \varepsilon_d{}^p]}{\sqrt{3}[1 + (\varepsilon_d{}^p)^2]}$$

Note : For IMOD=19, the parameter $\varepsilon_d{}^p$ is multiplied by 1000 (see Yuritzinn page 127).

Input parameters : A, B

• Modified Prevost and Hoeg hardening law: IECR = 3

$$k = \frac{A\,[B\,(\varepsilon_d{}^p)^2 + \varepsilon_d{}^p]}{\sqrt{3}[1 + (\varepsilon_d{}^p)^2]}$$

Note : The same model as IECR=2, without the multiplying factor 1000 on $\varepsilon_d{}^p$.

Input parameters : A, B

Note : regarding the use of « deviatoric hardening » see appendix (15.6). In the models above, the deviatoric strain is calculated by integrating $\dot{\lambda}\frac{\partial f}{\partial q} = \dot{\lambda}/\sqrt{3}$.

3.3.4.3. Coulomb criterion (ICRIT=3)

• Formulation : $f(\underline{\sigma}) = (\sigma_1-\sigma_3) - (\sigma_1+\sigma_3)\ \sin\varphi$

• Input parameters : φ

• **Hardening laws :** none

3.3.4.4. Mohr-Coulomb criterion (ICRIT=4)

• Formulation : $f(\underline{\sigma}) = (\sigma_1-\sigma_3) - (\sigma_1+\sigma_3)\ \sin\varphi - 2c\cos\varphi$

• Input parameters : c, φ

• **Hardening laws :** none

3.3.4.5. Mohr-Coulomb criterion with c and φ varying with depth (ICRIT=15)

• Formulation : $f(\underline{\sigma}) = (\sigma_1-\sigma_3) - (\sigma_1+\sigma_3)\ \sin\varphi - 2c\cos\varphi$

with $c = c_o + y\,\Delta c$; $\varphi = \varphi_o + y\,\Delta\varphi$

• Input parameters : c_o, Δc, φ_o, $\Delta\varphi$

NB : y represents the coordinate of the considered point along the upward vertical direction: if the strength is to increase with depth, the vertical gradient Δc must be negative. Please note also that the values of the strenght properties depend on the origin of the vertical coordinates.

- **Hardening laws:** none

3.3.4.6. Drucker Prager criterion (ICRIT=5)

- Formulation : $f(\underline{\sigma}) = \sqrt{J_2} + \alpha\, I_1 - k$

  with $I_1 = \sigma_1 + \sigma_2 + \sigma_3$; $J_2 = 1/2\ \underline{\underline{s}} : \underline{\underline{s}}$

- Input parameters : c, φ (le code computes α and k : $\alpha = \dfrac{\tan\varphi}{\sqrt{9+12\tan^2\varphi}}$; $k = \dfrac{3c}{\sqrt{9+12\tan^2\varphi}}$)

- **Hardening laws :**

<u>(Negative) hardening with constant hardening module : IECR=1</u>

$\delta k = -H\,\delta\lambda$ ($\delta\lambda$ = plastic multiplier)

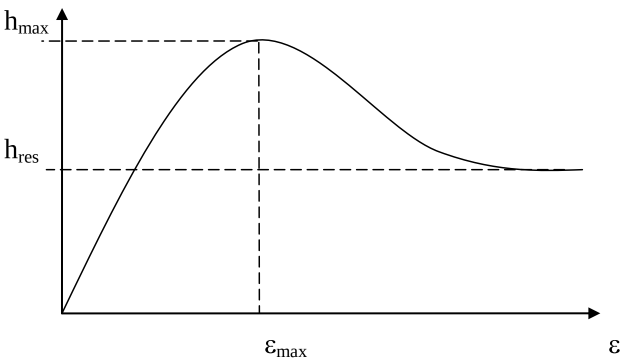
input parameter : H

Note : it is the same model as in IMOD=14 if $H = \mu / (1+\chi)$, where μ is le shear modulus (E/2(1+ν)) and χ the hardening parameter of model IMOD=14

<u>- variable volumetric hardening : IECR=3</u>

Hardening concerns both parameters α and k, equal to $(1+\eta)\,\alpha°$ and $(1+\eta)\,k°$, with $\eta = h(\varepsilon_p^v)$

$h(x) = \dfrac{ax+bx^2}{1+cx^2}$ where $a = 2\,h_{max}/x_{max}$; $c = \dfrac{h_{max}}{(h_{max} - h_{res})\,x_{max}^{\ 2}}$; $b = c\,h_{res}$



Input parameters : h_{max} ; h_{res} ; x_{max} (all dimensionless)

Note : the model is derived from the Prévost and Hoeg model.

<u>- variable volumetric hardening : IECR=4</u>

Hardening concerns both parameters α and k, equal to $(1+\eta)\,\alpha°$ and $(1+\eta)\,k°$, with $\eta = h(\varepsilon_p^v)$

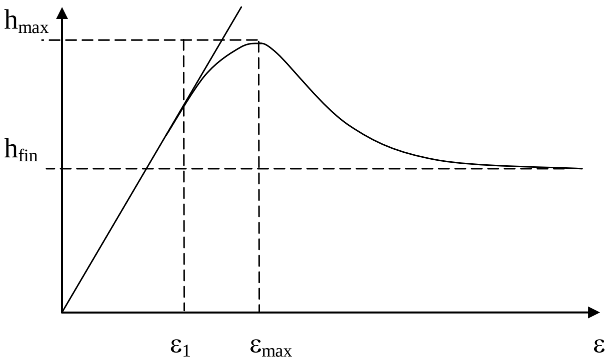
One computes $\varepsilon_1 = \frac{2\, h_{max}}{c} - \varepsilon_{max}$ and $\alpha = \frac{c^2}{4(c\varepsilon_{max} - h_{max})}$

if $x < \varepsilon_1$: $h(x) = c\, x$ $h'(x) = c$

if $\varepsilon_1 < x < \varepsilon_{max}$: $h(x) = c\, x - \alpha\, (x - \varepsilon_1)^2$ $h'(x) = c - 2\alpha\, (x - \varepsilon_1)$

if $x > \varepsilon_{max}$: $h(x) = h_{fin} + (h_{max} - h_{fin}) \exp\left[- \left(\frac{x-\varepsilon_m}{\beta}\right)^2 \right]$

$$h'(x) = - \frac{2(x-\varepsilon_m)}{\beta^2} (h_{max} - h_{fin}) \exp\left[- \left(\frac{x-\varepsilon_m}{\beta}\right)^2 \right] = - \frac{2(x-\varepsilon_m)}{\beta^2} (h - h_{fin})$$



Input parameters : c, h_{max} ; h_{fin} ; ε_{max} ; β (all are dimensionless parameters)

Notes :

1- this is a variant of the previous hardening law (IECR=3 for ICRIT=5), with two more parameters so that we can better represent the results of triaxial tests on real materials. In particular, we completely separate the two parts of the curve to the left and to the right of the peak (apart from the fact that the curve is continuous). The parameter β, in particular, controls the decay after the peak.

2- the input parameters must fulfill the condition : $2\, h_{max} < c\, \varepsilon_m$

(the code checks if this condition is fulfilled).

Hardeling module H : $H\, \lambda = - \frac{\partial f}{\partial p_c} \dot{p}_c = - \left(\frac{\partial f\, \partial \alpha}{\partial \alpha \partial \varepsilon_v{}^p} + \frac{\partial f\, \partial k}{\partial k \partial \varepsilon_v{}^p} \right) \lambda\ \mathrm{tr}\left(\frac{\partial g}{\partial \sigma}\right)$

$H = - \left(I_1 \alpha_o - k_o \right) h'(\varepsilon_v{}^p)\ \ \mathrm{tr}\left(\frac{\partial g}{\partial \sigma}\right)$

3- the model is presented for informative purpose. It could be improved to better fit the curve obtained in a triaxial shear test and the response to cyclic shear (by introducing a non linear kinematic hardening).

3.3.4.7. Criterion of the Modified Cam-Clay model (ICRIT=6)

- Formulation : $f(\underline{\sigma}) = q^2 - M^2\, p\, (p_c - p)$

  with $q = \sqrt{3\, J_2}$; $p = -I_1/3$; $p_c = \max(p_c{}^\circ, p_{cs})$

  p_{cs} is a parameter stored for each integration point and reflects the history of loading

• Input parameters : M, $p_c°$

• **Hardening laws :**

- Volumetric hardening of the Modified Cam Clay model : IECR=1

$$p_C = p_C° \exp\left(-\frac{(1+e_o)\,\varepsilon_v^p}{(\lambda-\kappa)}\right)$$

Input parameters : λ, κ, e_o

Note : The user has to supply λ, κ and eo but the law only depends on the value of the ratio (1 + eo) / (λ-κ). Moreover, when one associates this model with the nonlinear elasticity classically proposed for the Cam-Clay model (IELAS = 3 above), there is no verification that the values of κ and eo are identical for the elastic part and the hardening law (which are dealt with independently in the code). On the other hand, this law is identical to that of the IMOD = 18 model of the standard version.

Note 2 : The model is frequently formulated differently, in terms of compression index C_c and swelling index C_s rather than the coefficients λ and κ. The latter are related to isotropic compression test in the presentation of the Cam Clay model given by Wood (1990), whereas C_c and C_s are defined with respect to oedometric tests. It is generally admitted that

$C_c = 2,3\ \lambda$; $C_s = 2,3\ \kappa$

- Volumetric and deviatoric hardening : IECR=3

the introduction of a deviatoric hardening makes it possible to model the increase of the mean and deviatoric stresses p' and q along the critical state line in undrained condition (the model can however give a limit on the shear stress according to the value of β).

$$p_C = p_C° \exp\left(-\frac{(1+e_o)\,\varepsilon_v^p}{(\lambda-\kappa)} - \beta\,\varepsilon_d^p\right)$$

Input parameters : λ, κ, e_o, β

- Volumetric hardening with initial preconsolidation pressure varying with depth: IECR=6

This model corresponds to the same volumetric hardening as IECR=1, but takes into account a variation of the initial preconsolidation pressure (the parameter defining the size of the ellipse) which varies with depth according to :

$p_c^{ini} = \mathrm{Max}\,(p_c°, p_{cz} + Dp_c\ \ z)$

where z is the coordinate along the vertical upward direction, whereas parameters p_{cz} and Dp_c describe the variations of the preconsolidation pressure with depth. Pay attention to the sign of Dp_c : it must be negative to represent anincrease in p_c as depth increases.

The evolution of the hardening parameter is given by:

$$p_C = p_C^{ini} \exp\left(-\frac{(1+e_o)\,\varepsilon_v^p}{(\lambda-\kappa)}\right)$$

Input parameters : p_{cz} , $Dp_{c,}$ λ, κ, e_o

- prise en compte d'un rapport de surconsolidation OCR : IECR=7

This model corresponds to the same volumetric hardening as IECR=1, but takes into account an initial preconsolidation pressure (the parameter defining the size of the ellipse) defined by a given over-consolidation ratio (OCR) :

$p_c^{ini} = \mathrm{Max}\,(p_c°, \mathrm{OCR}\ p_c^{ref})$

where p_c^{ref} is the value of the preconsolidation pressure computed from the initial stress field through

$p_c^{ref} = q^2 / M^2 p + p$

If initial stresses are geostatic, near the surface p is zero. To avoit this difficulty p_c^{ref} is omputed only if p is larger than a minimum value ; near the surface, p_c is taken equal to the value of $p_c°$ provided by the user in the parameters for the criterion.

The evolution of the hardening parameter is given by:

$$p_c = p_c^{ini} \exp\left(-\frac{(1+e_o)\,\varepsilon_v^p}{(\lambda-\kappa)}\right)$$

Input parameters : OCR, λ, κ, e_o

3.3.4.8. Parabolic criterion (ICRIT=7)

- Formulation : $f(\underline{\sigma}) = J_2 + (R_c - R_t)\, I_1/3 - R_c\, R_t /3$
- Input parameters : R_c, R_t
- **Hardening laws :** none

See also 0.

3.3.4.9. Generalized Hoek and Brown criterion (ICRIT=14)

- Formulation : $f(\underline{\sigma}) = \left(\frac{\sigma_1-\sigma_3}{\sigma_c}\right)^{1/\alpha} - (m\ \sigma_3/\sigma_c + s)$
- Input parameters : σ_c, s, m, α
- **Hardening laws :** none

3.3.4.10. Directional criterion (ICRIT=11)

- Formulation : $|\underline{\tau}| < \sigma \tan \varphi + c$

  σ : normal stress acting on a facet normal to the unit vector $\underline{n}$

  $\underline{\tau}$ tangential stress acting on the same facet

  $\sigma = \underline{n} \cdot \underline{\sigma} \cdot \underline{n}\ ;\ |\underline{\tau}| = |\underline{\sigma} \cdot \underline{n} - \sigma \cdot \underline{n}|$

Note : the elastic domain is larger than that obtained with the Mohr-Coulomb model for the same values of c and φ, because the condition only concerns the facets oriented in a specified way. It is clearly an anisotropic model.

- Input parameters :

  en 2D : c, φ, α : angle between Ox and $\underline{n}$

  en 3D : c, φ, n_x, n_y, n_z (the user inputs the coordinates of $\underline{n}$)
- **Hardening laws :** none

3.3.4.11. Directional criterion for chevron-shaped fractures (ICRIT=12)

This model was developed by A. Pouya to take into account the role of fractures in the vicinity of an excavation in argillites (se also the elastic models IELAS=9 or 10 above). Locally, the strength is described by the same crtierion as in the previous model (ICRIT=11), but the orientation of the facets on which the normal and tangential stresses are computed changes according to the position of the point: the model accounts for the variations of the orientation of the fractures.

The model cannot be used in plane strain.

In axisymmetric condition, the axis of revolution is necessarily the vertical axis (r = 0) of the mesh and we simply give the angle α between the radial direction er and the normal to the isotropy plane tangent to the fractures. In three-dimensional condition, we give the three coordinates (x, y, z) of a point on the axis of revolution, two angles α and β defining an oriented unit vector of the axis u = (cos α cos β , sin α cos β, sin β), and a last angle γ to represent the opening of the cones. For a current point M, we calculate its projected P on the axis, and we denote by v the unit vector carried by PM. The normal to the plane tangent to the fractures is given by v [-sin γ] + u [cos γ].

- Formulation : $|\underline{\tau}| < \sigma \tan \varphi + c$

  σ : normal stress acting on a facet normal to the unit vector $\underline{n}$

  $\underline{\tau}$ tangential stress acting on the same facet

  $\sigma = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n}$; $|\underline{\tau}| = |\underline{\underline{\sigma}} \cdot \underline{n} - \sigma \cdot \underline{n}|$

- Input parameters :

  In axisymmetrical conditions:

  c, φ, α (α is the angle between the radial unit vector and $\underline{n}$)

  in 3D :

  c, φ,

  x, y, z (coordinates of a point on the axis of revolution),

  α, β (two angles defining the orientation of the axis),

  γ (cone opening angle)

- **Hardening laws :** none

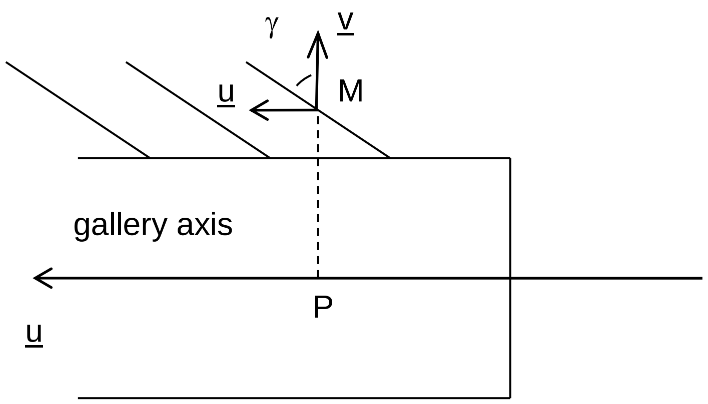


Figure 2 – Definition of the geometry of the chevron-shapes fractures

Note : for fractures oriented as in the figure (with the tip of the cones on the side of the gallery's working face), the angle γ takes a positive value. If we change the orientation of the gallery axis, the value of γ corresponding to the same geometry of fractures is negative.

(see datafiles: chevr_icr12.data and chvr3_icr12.data).

3.3.4.12. Hill criterion (ICRIT=22)

Hill proposed a model to account for anisotropic strength (1947). The formulation adopted here is excerpted from the user manual of the DIANA FEA software (quoted in the references):

$$f(\sigma) = \sqrt{\frac{3}{2}\sigma^T P \sigma} - \sigma_{ref},$$

where

σ denotes the vector formed by the components of the stress tensor

$$\sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})$$

σ_{ref} denotes a scalar reference stress value

and P denotes the matrix defined by :

$$P = \frac{1}{3}\begin{bmatrix} \alpha_{12}+\alpha_{13} & -\alpha_{12} & -\alpha_{13} & 0 & 0 & 0 \\ -\alpha_{12} & \alpha_{23}+\alpha_{12} & -\alpha_{23} & 0 & 0 & 0 \\ -\alpha_{13} & -\alpha_{23} & \alpha_{13}+\alpha_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\alpha_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 6\alpha_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 6\alpha_{66} \end{bmatrix}$$

with

$$\begin{bmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_{44} \\ \alpha_{55} \\ \alpha_{66} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2(\sigma_{ref}/\sigma_{y.xx})^2 \\ 2(\sigma_{ref}/\sigma_{y.yy})^2 \\ 2(\sigma_{ref}/\sigma_{y.zz})^2 \\ (\sigma_{ref}/\sigma_{y.xy})^2 \\ (\sigma_{ref}/\sigma_{y.yz})^2 \\ (\sigma_{ref}/\sigma_{y.zx})^2 \end{bmatrix}$$

Parameters $\sigma_{y.xx}$, $\sigma_{y.yy}$, $\sigma_{y.zz}$ represent the maximal stresses in pure tension in directions x, y and z attached to the material, and $\sigma_{y.xy}$, $\sigma_{y.yz}$, $\sigma_{y.zx}$ the maximum stresses in pure shear (the first index y meaning "yield").

Note 1 : For $\sigma_{y.xx} = \sigma_{y.yy} = \sigma_{y.zz} = k\sqrt{3}$ and $\bar{\sigma} = \sigma_{y.xy} = \sigma_{y.yz} = \sigma_{y.xz} = k$, we get : $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{44} = \alpha_{55} = \alpha_{66} = 1/3$ and

$$P = \frac{1}{9}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

In this situation the Hill criterion coincides with von Mises criterion.

Note 2 : in practice, in CESAR, the criterion is formulated as :

$f(\sigma)=3/2\,{}^t\sigma\ P\ \sigma - \sigma_{ref}{}^2$

so as to simplpify the computation of the derivatives of the criterion, by removing the square root.

- Formulation : $f(\underline{\sigma}) = 3/2\,{}^t\sigma\ P\ \sigma - \sigma_{ref}{}^2$
- Input parameters : in 2D : $\sigma_{y.xx}$, $\sigma_{y.yy}$, $\sigma_{y.zz}$, $\sigma_{y.xy}$, $\sigma_{y.yz}$, $\sigma_{y.xz}$, σ_{ref}, ψ
  in 3D : $\sigma_{y.xx}$, $\sigma_{y.yy}$, $\sigma_{y.zz}$, $\sigma_{y.xy}$, $\sigma_{y.yz}$, $\sigma_{y.xz}$, σ_{ref}, ψ, θ, φ
- **Hardening laws :** none

The parameters describing the orientation of the material are described in appendix 15.1.

3.3.4.13. Modified Hill criterion (ICRIT=23)

For some materials it can be difficult to determine the tensile strength in one direction (for instance in the direction perpendicular to a metallic plate).

It can be easier to give the strenghts in the plate plane, in an oblique direction, for instance the tensile strength in a direction of 45 degrees in the (x,y) plane. The criterion is defined by the same matrix P, but the coefficients are given by:

$$\begin{bmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{23} \\ \alpha_{44} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -2 & 1 \\ 1/2 & -1/2 & 2 & -1 \\ -1/2 & 1/2 & 2 & -1 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} . \begin{bmatrix} 2(\sigma_{ref}/\sigma_{y.xx})^2 \\ 2(\sigma_{ref}/\sigma_{y.yy})^2 \\ 2(\sigma_{ref}/\sigma_{y.45})^2 \\ (\sigma_{ref}/\sigma_{y.xy})^2 \end{bmatrix} ;\ \alpha_{55} = \alpha_{66} = 0$$

- Formulation : $f(\underline{\sigma}) = 3/2\,{}^t\sigma\ P\ \sigma - \sigma_{ref}{}^2$
- Input parameters : in 2D : $\sigma_{y.xx}$, $\sigma_{y.yy}$, $\sigma_{y.45}$, $\sigma_{y.xy}$, σ_{ref}, ψ
  in 3D : $\sigma_{y.xx}$, $\sigma_{y.yy}$, $\sigma_{y.45}$, $\sigma_{y.xy}$, σ_{ref}, ψ, θ, φ
- **Hardening laws :** none

The parameters describing the orientation of the material are described in appendix 15.1.

3.3.4.14. Hill-Lourenço criterion (ICRIT=28)

Another version of the Hill criterion has also been implemented in CESAR. It is based on the same general expression:

$f(\sigma)=3/2\,{}^t\sigma\ P\ \sigma - \sigma_{ref}{}^2$

with

$$P = \frac{2}{3}\begin{bmatrix} 1/\Sigma_{11}^2 & \beta & 0 & 0 & 0 & 0 \\ \beta & 1/\Sigma_{22}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\Sigma_{12}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The yield surface is an ellipsoid in the space of the stresses (σ_{11}, σ_{22}, σ_{12})

- Formulation : $f(\underline{\sigma}) = (\sigma_{11}/\Sigma_{11})^2 + (\sigma_{22}/\Sigma_{22})^2 + (\sigma_{12}/\Sigma_{12})^2 + \beta\ (\sigma_{11}\sigma_{22} / \Sigma_{11}\Sigma_{22}) - 1$
- Input parameters : in 2D : Σ_{11}, Σ_{22}, β, Σ_{12}, ψ

  in 3D : σ_{11}, σ_{22}, β, σ_{12}, ψ, θ, φ

- **Hardening laws :** none

The parameters describing the orientation of the material are described in appendix 15.1.

3.3.4.15. Hill Lourenço criterion variant (ICRIT=29)

It is a generalization of the previous criterion.

- Formulation : $f(\underline{\sigma}) = (\sigma_{11}/\Sigma_{11})^2 + (\sigma_{22}/\Sigma_{22})^2 + (\sigma_{33}/\Sigma_{33})^2$

  $+ (\sigma_{12}/\Sigma_{12})^2 + (\sigma_{23}/\Sigma_{23})^2 + (\sigma_{13}/\Sigma_{13})^2 + \beta\ (\sigma_{11}\sigma_{22} / \Sigma_{11}\Sigma_{22}) - 1$

- Input parameters : en 2D : Σ_{11}, Σ_{22}, Σ_{33}, β, Σ_{12}, Σ_{23}, Σ_{13}, ψ

  en 3D : Σ_{11}, Σ_{22}, Σ_{33}, β, Σ_{12}, Σ_{23}, Σ_{13}, ψ, θ, φ

- **Hardening laws :** none

The parameters describing the orientation of the material are described in appendix 15.1.

3.3.4.16. Menetrey and Willam model based criterion (ICRIT=16)

Menetrey and Willam have proposed, for concrete, a smooth yield surface that depends on the three invariants:

$F = (A\rho)^2 + m\ (B\ \rho\ r + C\ I_1/\sqrt{3}\ ) - c$

where

$\rho = \sqrt{2\ J_2}$, $I_1 = \text{tr}\ \underline{\sigma}$

A, m, B, C and c are material properties,

r depends on the Lode angle θ and on aen eccentricity parameter e :

$r(\theta,e) = [\ 4\ (1-e^2)\cos^2(\theta) + (2e-1)^2] \ /\ [\ 2\ (1-e^2)\cos\theta + (2e-1)\ \{4(1-e^2)\cos^2(\theta) + 5\ e^2 - 4^e\}^{1/2}]$

(it is recalled that : $\cos(3\theta) = 3\sqrt{3}\ \ J_3 / (2\ J_2^{3/2})$, where $J_2 = 1/2\ \underline{s} : \underline{s}$ and $J_3 = 1/3\ \text{tr}\ (\underline{s}^3)$)

The function $r(\theta,e)$ represents the trace of the criterion in the deviatoric plane. It is readily seen that, if e=1, the sitance between the failure surface and the isotropic compression axis depends on θ. For the elastic domain to be convex, e must be (strictly) largter than 0,5. For e=0,5, the yield surface is no longer smooth.

With an appropriate choice of the parameters, function F makes it possible to reproduce a range of models: Drucker Prager, Rankine, Huber-Mises. Notably, one can get a yield surface close to that of the Mohr Coulomb model, but with a smooth yield surface, by letting:

$A = 0$; $e = (3-\sin\varphi)/(3+\sin\varphi)$; $B = (3-\sin\varphi)/(\sqrt{24}\cos\varphi)$; $C = \tan(\varphi)/\sqrt{3}$; $m = 1$

C is the cohesion in the sense of the Mohr-Coulomb model.

The proposed function takes the same values at that of Mohr-Coulomb fall all stress states in which two of the principal stresses are equal (in other words, the surface includes the edges of the Mohr Coulomb surfaces). The proximity between the surfaces depends on φ : pour $\varphi = 0$, e=1 and the criterion becomes that of von Mises.

Note 1 : the complete formulation of the Menetrey Willam model has not been implemented yet (hardening, flow rule).

- Formulation : $f(\underline{\underline{\sigma}}) = (B\,\rho\, r(\theta,e) + C\, I_1/\sqrt{3}) - c$

  with $\rho = \sqrt{2J_2}$, $I_1 = \mathrm{tr}\,\underline{\underline{\sigma}}$, $B = \dfrac{3 - \sin\varphi}{\sqrt{24}\cos\varphi}$; $C = \dfrac{\tan\varphi}{\sqrt{3}}$
- Input parameters : c, φ
- **Hardening laws :** none

3.3.4.17. Bigoni-Piccolroaz model based criterion (ICRIT=17)

Bigoni and Piccolroaz have proposed a yield function that depends on the third invariant:

$F(\sigma) = f(p) + q\, g(\theta)$

where f(p) is called the « meridian function » defined by :

$f(p) = -M\, p_c\sqrt{(\Phi-\Phi^m)\,[2(1-\alpha)\,\Phi + \alpha\,]}$ si $\Phi \in [0,1]$ and $f(p)=+\infty$ sinon

where $\Phi = (p+c)/(p_c+p)$.

The « deviatoric function » $g(\theta)$ is given by:

$g(\theta) = \cos\left(\beta\pi/6 - 1/3 \arccos(\gamma\cos(3\theta))\right)$

The model includes 7 parameters, which are bound to fulfill the following conditions:

$p_c>0$, $\alpha\in\ ]0,2[$, $m>1$, $M>0$, $\beta\in [0,2]$ and $\gamma\in[0,1[$.

If such is the case, the yield surface is defined and smooth.

According to the value of β, the yield surface in the deviatoric plane is more or less close to an hexagon. It γ is chosen very close to 1, the surface becomes less smooth.

To obtain a yield surface close to Mohr-Coulomb, the authors propose to take:

$\alpha = 0$; $p_c=f_c$ (compression strength) $= 2c\cos\varphi/(1-\sin\varphi)$; $M = 3\,[r\cos(\beta\pi/6 -\pi/3) - \cos(\beta\pi/6)] / [\sqrt{2}\,(r+1)$

$c = fc\,[\cos(\beta\pi/6 -\pi/3) - \cos(\beta\pi/6)] / [3r\cos(\beta\pi/6 -\pi/3) - 3\cos(\beta\pi/6)]$

$r = f_t/f_c = 1-\sin\varphi/(1+\sin\varphi)$; $\beta\pi/6 = \arctan(\sqrt{3}/(2r+1))$; $\gamma\rightarrow 1$; $m\rightarrow\infty$

- Input parameters : c, φ, γ, n

- **Hardening laws :** none

Note : For the numerical implementation in CESAR, this criterion requires a treatment very similar to that of the previous model. Note however that if γ is very close to 1, numerical instabilities can occur in the vicinity of the edges of the yield surface: in such cases, it is recommended to monitor carefully the influence of the load increments.

As for the previous model, it could be fruitful to introduce the flow rule proposed by Bigoni and Piccolroaz for the materials like concrete. However, the mathematical formulation is a bit complex. It is also possible that the Hiss model provides the same advantages as the Menetrey-Willam and Bigoni-Piccoloraz with less complexity.

3.3.4.18. Critère du modèle Egg Cam Clay (ICRIT=19)

This is a variant of the modified Cam Clay model with two main differences:

- the yield surface is shifted to allow tensile stresses,
- the shape of the surface is modified to have a flattened shape in the frictional part (above the critical state line).

The model also makes it possible to modifiy the hardening law so as to prevent the yield surface to shrink.

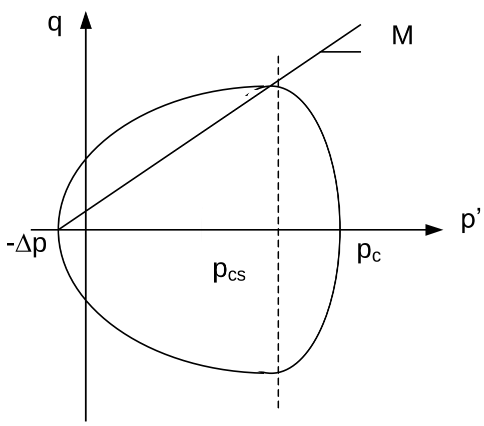


Figure 3 – Definition of the yield surface of the Egg Cam Clay model

p_{cs} is chosen in such a way that the ratio between the horizontal semi-axes of the half-ellipses is constant and equal to α : $(p_c - p_{cs}) = \alpha\,(p_{cs} + \Delta p)$ $(\alpha<1)$

- Formulation : $f(\underline{\sigma}) = q^2 + M^2\,(p+\Delta p)(p-\Delta p-2p_{cs})$ si $p<p_{cs}$

  $f(\underline{\sigma}) = q^2 + M^2\,(p+p_c-2p_{cs})\,(p-p_c)$ si $p\geq p_{cs}$

- Input parameters : M, pc_o, Δp, α

- Hardening laws :

  Volumetric hardening of the Cam-Clay model : IECR=1 (see ICRIT=6).

$$p_C = p_C{}^{\circ} \exp\left(-\frac{(1+e_o)\ \varepsilon_v{}^p}{(\lambda-\kappa)}\right)$$

Input parameters: λ, κ, e_o

- hardening in compression only: IECR=2

$dp_C=0$ si $p<p_{cs}$

$dp_C = p_C\left(-\frac{1+e_o}{\lambda-\kappa}\right) d\varepsilon_v{}^p$ si $p>p_{cs}$

Input parameters : λ, κ, e_o

3.3.4.19. Critère du modèle HiSS (ICRIT=24)

This criterion is part of a model proposed by Shao and Desai (2000), called Hierarchical Single Surface model, for the cyclic and dynamic behaviour of saturated soils. Its yield surface depends on the third invariant of the stress tensor, and combines a "frictional" part (close to Mohr Coulomb or Drucker Prager) and a « contractive » part (similar to the Cam Clay model). The third invariant is introduced in a simpler way than in the Menetry-Willam or Bigoni-Piccolroaz criteria.

- Formulation : $f(\underline{\sigma}) = \frac{J_{2d}}{p_a{}^2} - [\gamma\ (\frac{J_1^*}{p_a})^2 - \alpha(\frac{J_1^*}{p_a})^n] / \sqrt{1-\beta S}$

  avec $n>2$, $S = \frac{\sqrt{27}}{2}\frac{J_3}{J_2{}^{3/2}}$, $J_1^* = \mathrm{tr}\ \underline{\sigma} + 3R$

- Input parameters : γ β n R α_0 p_a

Note : S varies between -1 and 1 (the extremal values correspond to the stress states for which two principal stresses are equal). On the vicinity of the apex, the opening of the surface is controlled by parameter γ (analogous to the α factor of the Drucker-Prager if β is set to zero). Parameter β controls the ratio between the compression and tne extension limits. Parameter α of the Hiss model defines the size of the yield surface and can be taken as hardening parameter.

- Hardening laws :

  <u>- Volumetric hardening: IECR=1</u>

  $\alpha = \alpha_0 \exp(-\lambda\ \varepsilon_v^p)$

  Input parameters : λ

  <u>- Volumetric and deviatoric hardening: IECR=2</u>

  $\alpha = \alpha_0 \exp(-\lambda\varepsilon_v^p - \mu\varepsilon_d^p)$

  Input parameters : λ, μ

ε_d^p is obtained by integrating $\dot{\xi}_d^p = (\frac{2}{3}\ \dot{\underline{\varepsilon}}_d^p : \dot{\underline{\varepsilon}}_d^p)^{1/2}$

3.3.4.20. Modified HiSS criterion (ICRIT=30)

The formulation can be reworked to clarify the role of the parameters controlling the yield surface.

- Formulation : $f(\underline{\underline{\sigma}}) = J_2 - \gamma\, p^2 \,[\,1 - (p/p_c)^m]\,/\,\sqrt{1-\beta S}$

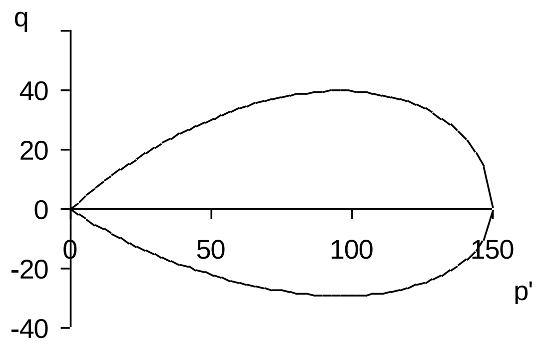
  with $p = -\dfrac{1}{3}\,tr\,(\underline{\underline{\sigma}}) + R$; $m > 0$; $S = \dfrac{\sqrt{27}}{2}\,\dfrac{J_3}{J_2^{3/2}}$

- input parameters : γ m $p_c°$ (initial value of p_c) R β
- Hardening laws:

  - volumetric hardening : IECR=1

  $p_c = p_c°\,\exp\,(-A\varepsilon_v^p)$

  input parameters : A



Valeurs des paramètres :

p_c=150, β = 0,3 ; γ = 0,2 ; m =0,5

Figure 4 – Trace of the yield surface of the Hiss model for S=1 et S=-1.

Notes :

The elastic domain is limited, on the isotropic stres state axis, by p=0 and $p = p_c$.

The parameter m controls the shape of the cap on the isotropic compression side: larger values of m correspond to steeper yield surfaces. For excessively large values of m, the yield surface becomes less smooth, and the numerical treatment is more difficult (possible convergence issues).

For β= 0, in the vicinity of the apex, the term in p^m becomes negligible (<< 1), and the slope of the q-p curve tends to $\sqrt{3\gamma}$ (which makes it possible to adjust the value of γ to get a yield surface cose to that of the Drucker Prager model).

The critical states (dq/dp = 0) correspond to $\dfrac{p}{p_c} = \left[\dfrac{2}{m+2}\right]^{1/m}$

They belong to a critical state line of slope:

$$\sqrt{3\gamma\,\frac{m}{m+2}}$$

Note : if the third invariant is taken into account, near the apex, the surface is close to the Mohr-Coulomb one, in triaxial compression as well as in triaxial extension by fitting the values of β and γ :

$$J_2 = q^2/3 = \frac{\gamma\, p^2}{\sqrt{1-\beta S}}$$

with S = -1 in compression and S = +1 in extension.

For Mohr Coulomb, failure corresponds to $q = \dfrac{6 \sin \varphi}{3 + \beta \sin \varphi} p$

Which leads to the following choice: $\beta = \dfrac{\alpha - 1}{\alpha + 1}$ with $\alpha = \dfrac{3 + \sin \varphi}{3 - \sin \varphi}$; and $\gamma = \dfrac{\sqrt{1-\beta}}{3}\left(\dfrac{6 \sin \varphi}{3 - \sin \varphi}\right)^2$

Moreover, one can adopt R = c cotan φ.

3.3.4.21. S-Clay 1 model (ICRIT=26)

The S-Clay 1 model was proposed by Wheeler et al (2003) to account for the anisotropy of strength properties of clays induced by the load history. It is a complex model, described in detail in appendix 15.2. It seems rich but difficult to use.

- Formulation : $3/2\ (\underline{\underline{s}} - p\ \underline{\underline{a}}^*) : (\underline{\underline{s}} - p\ \underline{\underline{a}}^*) - (M^2 - 3/2\ \underline{\underline{a}}^* : \underline{\underline{a}}^*)\ (p_m - p)\ p$
- input parameters : M, p_{m0}, k_0

The parameter k_0 is used for the initialiazation of $\underline{\underline{a}}^*$. see appendix 15.2

- Hardening laws :
  - Hardening law of the S-Clay 1 IECR=1

    The model describes the evolutions of p_m and of the tensorial variable $\underline{\underline{a}}^*$.

$$p_m = p_{m0} \exp\left[-\frac{1+e_0}{\lambda - \kappa}\varepsilon_v\right]$$

$$\alpha_0 = \frac{\eta_{Ko}^2 + 3\eta_{Ko} - M^2}{3}, \quad \eta_{Ko} = \frac{3(1 - K_o)}{1 + 2 K_o}$$

$$d\alpha = \mu\, [\, \{ \chi_v - \alpha \} < d\varepsilon_v^p > + \beta \{ \chi_d - \alpha \}\, d\varepsilon_d^p\,]$$

$$\chi_v = \frac{3\, \underline{\underline{s}}}{4\, p}\ ; \chi_d = \frac{\underline{\underline{s}}}{3\, p}\ ; \quad < d\varepsilon_v^p > = d\varepsilon_v^p \text{ if } d\varepsilon_v^p < 0 \text{ and 0 otherwise}$$

$$\beta = \frac{3\,(4 M^2 - 4\, \eta_{Ko}^2 - 3\eta_{Ko})}{8\, (\eta_{Ko}^2 + 2\eta_{Ko} - M^2)}$$

- Input parameters : λ, κ, e_0, μ

3.3.5. Simplified modelling of reinforcement inclusions

The classical model IMOD=43 makes it possible to take into account, by meands of a homogeniation approach, the influence of reinforcement inclusions in a ground mass whose behaviour is described by the Drucker Prager model. The same approach can be applied to any type of mechanical behavior for the ground, in the framework of the suer defined models. It is also possible to adopt more complex behaviours for the reinforcement inclusions themselves.

Note that the model relies on the assumption of a perfect bonding between the ground and the inclusions.

In the proposed structure, the user inputs, after the lines describing the behaviour of the ground, the following lines, associated with the reinforcement inclusions mechanical properties and to their geometrical arrangement:

RENF

Integer Indicator ICOMP for the mechanical properties of the inclusions followed by a list of parameters

Integer Indicator IGEOM for the geometrical arrangement of the inclusions followed by a list of parameters

3.3.5.1. Constitutive model for the reinforcement inclusions

The mechanical properties of the « phase » that represents the inclusions is described by an indicator called ICOMP. Two cases can be considered:

- ICOMP=1 corresponds to linear elasticity. Two parameters are required: the Young's modulus of the material the inclusions are made of and the cross sectional area of one inclusion;

- ICOMP=2 corresponds to a linear-elastic perfectly plastic behaviour: in addition to the previous parameters, input of the tensile strength of the material contitutive of the inclusions, and of the ratio between the compressive strength to the tensile strength (this ratio makes it possible to forbid compression in the inclusions).

Two other models are available but not entirely tested yet

- ICOMP=3 corresponds to an elastic-fragile behaviour, which results in the transfer of the load from the inclusions to the ground if the tensile stress in the inclusions reaches a threshold value; ICOMP=4 is a model in which three phases can take place : linear elastic, perfectly plastic, fragile failure it the plastic strain exceeds a given value.

The fragile models remain to be validated (the load transfers may lead to numerical issues).

ICOMP	description	input parameters
1	Linear elasticity	K : Young's modulus of the inclusions SA : cross sectional area of an inclusion
2	Linear elasticity + perfect plasticity	K, SA : same as for ICOMP=1, S : tensile strength of the material of the inclusions ETA : ratio of the compressive strength to the tensile strength
3	Linear elasticity + fragile failure	K, SA : same as for ICOMP=1 S, ETA : same as for ICOMP=2
4	Linear elasticity + perfect plasticity+ fragile failure	K, SA same as for ICOMP=1, S, ETA : same as for ICOMP=2 DPMT, DPMC : maximal values of the plastic strains in extension and in compression

3.3.5.2. Geometrical arrangement of the inclusions

The geometrical arrangement is described by an indicator called IGEOM. The input parameters depend on the values of NDIM (problem dimension) and INAT (for bidimensional problems). The parameters are the same as for the "classical" model IMOD=43 (cf. 3.2.20).

For NDIM= 2 ; INAT = 1

IGEOM	name	input parameters
1	Homogenous reinforcement	α, FV
2	Radial reinforcement	X, Y, FV

For NDIM=2 ;INAT = 2

valeur de IGEOM	name	input parameters
1	Homogenous reinforcement	FV
2	Radial reinforcement	FV
3	Cylindrically diverging reinforcement	α, FV
4	Spherically diverging reinforcement	Z, FV

For NDIM=3

valeur de IGEOM	name	input parameters
1	Homogenous reinforcement	α, β, FV
2	Radial reinforcement	X, Y, Z, α, β, FV
3	Cylindrically diverging reinforcement	X, Y, Z, α, β, γ, FV
4	Spherically diverging reinforcement	X,Y,Z, FV

3.3.6. Undrained behaviour

It is possible to specify that a group of elements is subjected to an undrained evolution by adding in the data relative to this group the following line

```
NDR    Kw   n
```

where K_w is the bulk modulus of water (around 2 GPa for pure water, but it can be more appropriate to use a smaller value to avoid numerical difficulties and/or to account for the increased compressibility of water containing dissolved air) ; n is the porosity.

Note : The values of n and Kw need not to be known with a great precision ; the only relevant information is that Kw/n must be sufficiently larger than the drained bulk modulus (typically 20 times).

This functionality leads to the computations of increments of the pore pressure. The use of the option PRS of the module MCNL males it possible to reconstitute a distribution of hydraulic head for a subsequent flow calculation.

3.4. Multi-mechanisms models

The keywords CRT2, POT2, ECR2 make it possible to define a second plastic mechanism (as a combination of a criterion, a plastic potential and a flow rule). The same functions can be used as criteria and potentials as for the first plastic mechanism. A small number of two-mechanism model have been implemented (to be completed).

3.5. Cyclic behaviour

For cyclic behaviour, one possibility is to introduce a non linear kinematic hardening. In CESAR, this has been done in a preliminary way by introducing specific values of ICRIT, but actually, one could have completed the existing hardening models for ICRIT=2 and ICRIT=5.

Two models have been proposed that make it possible to account for a progressive accumulation of plastic strains during repeated loadings of constant amplitude.

3.5.1.1. von Mises criterion for cyclic behaviour (ICRIT=25)

- Formulation : $f(\underline{\underline{\sigma}}) = F(\underline{\underline{\sigma}}-\underline{\underline{X}})$ where $F(\underline{\underline{\sigma}}) = \frac{1}{2}\,\underline{\underline{s}}:\underline{\underline{s}} - k^2$
- Input parameters : k
- Hardening law :

  - Chaboche (or Armstrong-Frederick) model : IECR = 1

    $\dot{\underline{\underline{X}}} = 2/3\ C\ \dot{\underline{\underline{\varepsilon}}}^p - D\ \underline{\underline{X}}\ \dot{\xi}$ with : $\dot{\xi} = (2/3\ \dot{\underline{\underline{\varepsilon}}}^p : \dot{\underline{\underline{\varepsilon}}}^p)^{1/2}$

    input parameters : C, D

Note : the implementation remains to be modified to let the user specify the initial value of X.

3.5.1.2. Drucker Prager criterion for cyclic behaviour (ICRIT=27)

- Formulation : $f(\underline{\underline{\sigma}}) = F(\underline{\underline{\sigma}}-\underline{\underline{X}})$ où $F(\underline{\underline{\sigma}}) = \sqrt{J_2} + \alpha\ I_1 - k$
- Input parameters : c, φ
- Hardening law :

  - Chaboche (or Armstrong-Frederick) model: IECR = 1

    $\dot{\underline{\underline{X}}} = 2/3\ C\ \dot{\underline{\underline{\varepsilon}}}^p - D\ \underline{\underline{X}}\ \dot{\xi}$ with : $\dot{\xi} = (2/3\ \dot{\underline{\underline{\varepsilon}}}^p : \dot{\underline{\underline{\varepsilon}}}^p)^{1/2}$

    input parameters : C, D

Note : ICRIT=25 with the proposed hardening law is relatively classic. The combination of Drucker Prager criterion with Cahboche model is relatively less frequent.

3.6. Hardening Soil Model (Frictional mechanism only)

3.6.1. Introduction

The Hardening Soil Model is an elastoplastic model with hardening for soils. Several (slightly different) presentations can be found in Schanz et al (1999), or in the documentation of the Plaxis software, which contributed to promote and spread the model.

The elastic part of the model is non linear and the models comprises two plastic mechanisms with hardening: a frictional (or deviatoric) mechanism, with a yield surface not too different from that of the Mohr Coulomb model, and contractive (isotropic) mechanism, which accounts for a contractive plastic strain is the mean stress becomes large enough.

In CESAR, only the deviatoric mechanism has been implemented.

3.6.2. Elastic part of the model (IELAS=5)

It has been described in section 3.3.3.13. It is an isotropic non linear model, with a constant Poisson's ratio, and a Young's modulus that depends on the minor principal stress:

$$E_{ur} = E_{ur}^{ref} \left(\frac{\sigma_3 + c \text{ cotan } \varphi}{\sigma_{ref} + c \text{ cotan } \varphi} \right)^m$$

3.6.3. Critère de plasticité et loi d'écrouissage (ICRIT=31 / ECR=1)

The yield surface is not smooth, and depends, like the Mohr Coulomb criterion, on the extreme principal stresses, which results in the fact that the surface has edges.

$$p = -1/3 \text{ tr } \sigma \quad ; \quad q_a = \frac{q_f}{R_f}$$

$$E_{50} = E_{50}^{ref} \left(\frac{\sigma_3 \sin \varphi_p + c \cos \varphi_p}{p_{ref} \sin \varphi_p + c \cos \varphi_p} \right)^m$$

$$q_f = \frac{2 (\sigma_3 \sin \varphi_p + c \cos \varphi_p)}{1 - \sin \varphi_p}$$

One of the criteria in the literature is:

$$f_{12} = \frac{2 \, q \, q_a}{E_i(q_a - q)} - 2 \frac{q}{E_{ur}} - \gamma^p \qquad \text{with } E_i = \frac{2 \, E_{50}}{2 - R_f}$$

ar, if $q = \sigma_1 - \sigma_2$:

$$f_{12} = \frac{q_a(2-R_f)}{E_{50}} \frac{\sigma_1 - \sigma_2}{q_a - (\sigma_1 - \sigma_2)} - 2 \frac{\sigma_1 - \sigma_2}{E_{ur}} - \gamma^p$$

In an analogous way, we let :

$$f_{13} = \frac{q_a(2-R_f)}{E_{50}} \frac{\sigma_1 - \sigma_3}{q_a - (\sigma_1 - \sigma_3)} - 2 \frac{\sigma_1 - \sigma_3}{E_{ur}} - \gamma^p$$

And the elastic domain corresponds to stress states for which both conditions $f_{12} < 0$ and $f_{13} < 0$ are fulfilled.

The model involves 7 parameters: c, φ_p, m, σ_{ref}, R_f, E_{ur}^{ref}, E_{50}^{ref}

Flow rule

Let

$$\sin \varphi_{cv} = \frac{\sin \varphi_p - \sin \psi_p}{1 - \sin \varphi_p \ \sin \psi_p}$$

$$\sin \varphi_m = \frac{\sigma_1 - \sigma_3}{\sigma_1 + \sigma_3 - 2 \ c \ \mathrm{cotan} \ \varphi_p}$$

$$\sin \psi_m = \frac{\sin \varphi_m - \sin \varphi_{cv}}{1 - \sin \varphi_m \ \sin \varphi_{cv}}$$

The plastic potential for f_{12} i :

$g_{12} = (\sigma_1 - \sigma_2)/2 - (\sigma_1 + \sigma_2)/2 \sin \psi_m$

The dilatancy angle ψ_m depends on σ_3 via φ_m. This should be taken into account to compute the derivatives of g_{12} with respect to σ, but it seems that this is neglected in the HSM (see formula 15 in Schanz et al, 2009). ON the other hand, the manua of Plaxis-GiD specifies a more complex rule to compute ψ_m :

$$\begin{array}{ll} \text{For } \sin\varphi_m < 3/4 \sin\varphi : & \psi_m = 0 \\ \text{For } \sin\varphi_m \geq 3/4 \sin\varphi \text{ and } \psi > 0 & \sin\psi_m = \max\left(\dfrac{\sin\varphi_m - \sin\varphi_{cv}}{1 - \sin\varphi_m \sin\varphi_{cv}}, 0\right) \\ \text{For } \sin\varphi_m \geq 3/4 \sin\varphi \text{ and } \psi \leq 0 & \psi_m = \psi \\ \text{If } \varphi = 0 & \psi_m = 0 \end{array} \qquad (4.12)$$

where φ_{cv} is the critical state friction angle, being a material constant independent of density, and φ_m is the mobilised friction angle:

$$\sin\varphi_m = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3 - 2c\cot\varphi} \qquad (4.13)$$

In the last place, to avoid excessive plastic dilatation, the user of Plaxis can define a threshold on the void ratio beyond which the dilatancy angle vanishes. This feature introduces 3 additional parameters (the initial value e_o of the void ratio and its minimal and maximal values e_{min} and e_{max}. It has not been implemented in CESAR.

Hardening law

The evolution of the parameter γ^p is defined (Schanz et al, 1999) by:

$d\gamma^p = d\varepsilon_v^p / \sin \psi_m$

3.6.4. Use of the HSM in CESAR

A specific treatment has been implemented for the plastic part of the HSM : the choice of the criterion entails the use of the specific flow rule above, and the user does not have to provide a POT keyword (in spite of the fact that the flow rule is non-associated). Also, the model is supposed to use the specific hardening law of the original model, so that the user only gives the keyword ECR followed by the indicator 1, without other parameters.

3.6.5. Limits of the Hardening Soil Model in CESAR

The implementation does not include the second plastic mechanism and the modelling of preconsolidation of the ground.

d

3.7. Damage models

3.7.1. Overview

As mentioned previously (cf. 2.7), damage models aim to account for a progressive loss of stiffness of a material as it is deformed. There is an abundant literature on the subject, and the modeling of these phenomena is the subject of numerous researches, which investigate the influence of the multiaxial character of the loading, or the restoration of rigidity when the cracks are closed, the anisotropic character of the loss of stiffness etc.

We first take the simplest point of view, that of isotropic damage. In this context, the tensor of the elasticity moduli (secant) is multiplied by a scalar less than 1 when the state of damage of the material changes.

The damage models have been developed to account for the non-linear behavior of certain "quasi-brittle" materials, which exhibit linear behavior up to a certain threshold (deformation or stress), then a loss of rigidity. The presentation given below is deliberately very succinct. It is based in particular on the syntheses proposed by Nedjar (1995) and Dal Pont and Tailhan (2009).

One of the first steps in taking damage into account is due to Kachanov (1958), who proposes to describe the evolution of elastic properties by a damage variable. In this approach, the stress tensor is linked to deformations by the following constitutive model :

$$\sigma = (1\text{-}D)\ C : \varepsilon$$

where C is the tensor of the elastic modules and d denotes the damage (scalar) variable: d is 0 for the undamaged material and 1 for the completely damaged material.

This approach was then developed by Lemaître and Chaboche (1985) in a thermodynamic framework, to study the damage of metals under the effect of the appearance of cavities and then their coalescence.

The continuation of the construction of the model consists in defining under which condition the damage variable evolves, and, if necessary, in giving the law describing this evolution. We note a certain similarity with the theory of plasticity, in which we define a plasticity threshold and a flow rule to describe the evolution of plastic strains.

In the above approach, the damage is represented by a single scalar variable. There are more complex damage models, in which damage is represented by a tensor variable to account for anisotropy. In the context of our study, we choose, at least in the first step, to stick to a scalar variable.

On the other hand, it is known that beyond a certain level of damage, a softening behavior is observed, which leads to a loss of uniqueness of the solution (see for example Jirasek, 2002, or Giry, 2011). To overcome this difficulty, different approaches are possible. We limit ourselves here to a local approach, the dependence on the mesh being managed by a simplified regularization technique of the Hillerborg type (that is to say that the digital processing of local behavior explicitly takes into account the size of the elements).

The most well-known isotropic damage model is that of Mazars, but many other models are available in the literature (for example that of Oliver).

In CESAR, the damage models have been programmed as part of the "user-defined" models (IMOD = 10000). Damage is associated with a specific keyword, ENDO, followed by an integer indicator IENDO, which represents the damage model chosen, and by a number of parameters which depends on the model adopted.

The rest of this section is partly borrowed from Moreno Regan (2016).

3.7.2. Implementation of damage models in the datafile

Damage models have been introduced in the MCNL module which gathers the main part of the nonlinear mechanical models in statics.

The principle consists in using the "user-defined" constitutive models (IMOD=10000). One then defines a linear isotropic elastic law using the keyword ELAS, then the damage law considered, using the keyword ENDO, followed by an integer indicator called IENDO which designates the chosen model and the appropriate parameters.

For instance, for a concrete whose initial characteristics are E = 10000 MPa and ν = 0.2, and for the Mazars model, with $\varepsilon_D{}^\circ = 10^{-4}$, Ac = 1; Bc = 1000; At = 1; Bt = 10,000; the following lines are found in the datafile:

`Beton`	name of the group of elements
`10000 1`	values of IMOD and INAT (plane strain)
`ELAS 0 10000.e6 0.2`	keyword ELAS and elastic parameters
`ENDO 6 1.e-4 1. 1000. 1. 10000.`	keyword ENDO + indicator IENDO for the damage model
	And damage parameters

3.7.3. Isotropic damage models

3.7.3.1. Mazars model (IENDO=6)

A class of models proposes to relate the damage criterion to a function of the tensor of deformations (or stresses).

Mazars (1984) proposed an isotropic damage model for concrete, based on experimental results. The evolution of the damage variable depends on an equivalent deformation $\tilde{\varepsilon}$ which expresses the local state of extension of the material: as long as it remains lower than a certain threshold, the material does not get damaged and its modules do not vary. If the threshold is reached, the damage variable d evolves according to a damage law that the model must specify.

In the original Mazars model, the equivalent deformation is given by:

$$\tilde{\varepsilon} = \sqrt{\langle\varepsilon_1\rangle_+^2 + \langle\varepsilon_2\rangle_+^2 + \langle\varepsilon_3\rangle_+^2}$$

where ε_i is the principal strain in direction i and where $\langle\varepsilon_i\rangle_+$ is given by:

$$\langle\varepsilon_i\rangle_+ = \varepsilon_i \quad \text{si} \quad \varepsilon_i \geq 0\,;\, \langle\varepsilon_i\rangle_+ = 0 \quad \text{si} \quad \varepsilon_i < 0$$

Damage is triggered when $\tilde{\varepsilon}$˜ exceeds a certain thresholdε_{D0}. Mazars' model (1984) proposes that this threshold be the strain corresponding to the maximum stress during a uniaxial direct traction test. If one supposes that the behavior is linear up to the maximum tensile stress, one can write:

$$r^0 = \frac{\sigma_t}{E_0}$$

where σ_t is the tensile strength and E_0 the modulus of elasticity prior to damage. The damage criterion is then written:

$$F = \tau - r \leq 0$$

where r is the current threshold, equal to the initial threshold ε_{D0} if it has never been reached, or to the largest value reached by τ otherwise. The limit surface of damage, defined by F = 0, is shown in the following figure by way of illustration, σ_c representing the compressive strength.

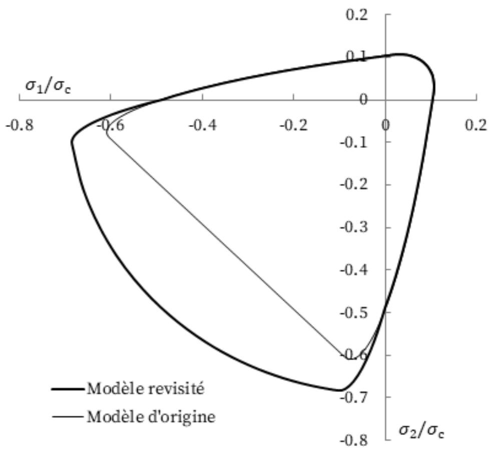


Figure – Limit surface for the Mazars damage model in the principal stresses space with $\sigma_3 = 0$

It remains to specify the evolution of the damage variable. Since the effect of the damage is not the same in tension and compression, Mazars (1984) considers that, for a state of multiaxial stresses, the variable d is a linear combination of two variables of damage associated respectively tensile stress d_t and compression stress d_c:

$$d = \alpha_t d_t + \alpha_c d_c$$

If there is no extension $\alpha_t = 0$; if there is no shrinkage $\alpha_c = 0$, and in all cases $\alpha_t + \alpha_c = 1$. The evolution laws for the two damage variables are:

$$d_c = 1 - \frac{r^0(1 - A_c)}{\tau_M} - \frac{A_c}{\exp[B_c - (r - r^0)]}$$

$$d_t = 1 - \frac{r^0(1 - A_t)}{\tau_M} - \frac{A_t}{\exp[B_t - (r - r^0)]}$$

where r^0 denotes the threshold deformation, determined experimentally and r the current value of the threshold. The parameters A_c and B_c (respectively A_t and B_t) are determined experimentally from the stress - strain curves of a compression test (respectively of traction).

The factors α_i which determine the contribution of traction or compression are determined according to the local state of deformations:

$$\alpha_t = \sum_{i=1}^{3} H_i \frac{\varepsilon_{ti}(\varepsilon_{ti} + \varepsilon_{ti})}{\tilde{\varepsilon}^2}$$

$$\alpha_c = \sum_{i=1}^{3} H_i \frac{\varepsilon_{ci}(\varepsilon_{ti} + \varepsilon_{ti})}{\tilde{\varepsilon}^2}$$

where $\tilde{\varepsilon}$ is the equivalent strain, and where $H_i = 0$ if $\varepsilon_i < 0$and $H_i = 1$ if $\varepsilon_i > 0$. The parts ε_{ti} and ε_{ci}correspond to the principal deformations caused by the stresses of tension and compression respectively. One calculates them by decomposing the tensor of effective principal stresses into a positive part and a negative:

$$\varepsilon_{ti} = s_{ij}^0 \bar{\sigma}_i^{\,+}$$

$$\varepsilon_{ci} = s_{ij}^0 \bar{\sigma}_i^{\,-}$$

In these expressions, if $\bar{\sigma}_i > 0$ then $\bar{\sigma}_i^{\,+} = \bar{\sigma}_i$, and $\bar{\sigma}_i^{\,-} = 0$; in the opposite case $\bar{\sigma}_i^{\,+} = 0$ and $\bar{\sigma}_i^{\,-} = \bar{\sigma}_i$. The complinace matrix prior to damage is calculated as follows:

$$s_{ij}^0 = \frac{1}{E_0}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}$$

The matrix s_{ij}^0 is used in plane strain or plane stress computations. Note also that:

$$\varepsilon_i = \varepsilon_{ti} + \varepsilon_{ci}$$

The type of stress-strain curve for this model is shown in the figure below.

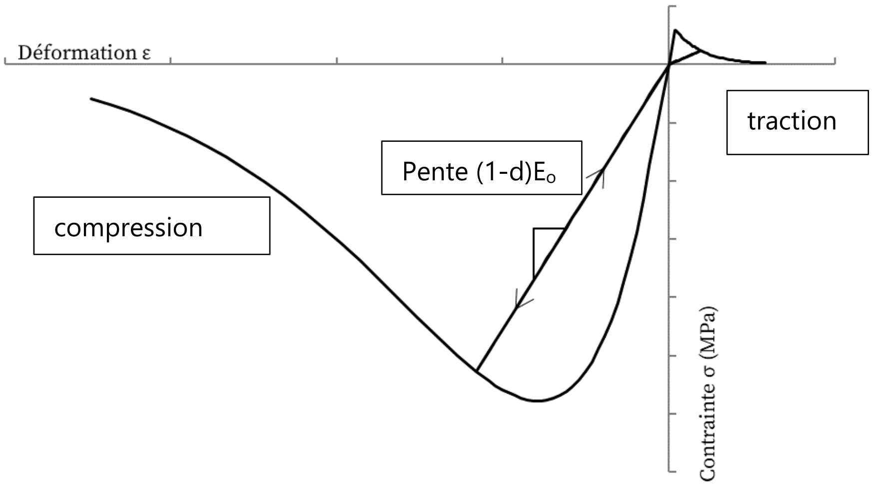


Figure – Uniaxial behaviour for the Mazars model

Parameters for the damage model Mazars (IENDO=6):

- strain defining the initial damage threshold $\varepsilon_D°$ [-]
- A_t, B_t : parameters for the damage law in traction [-]
- A_c, B_c : parameters for the damage law in compression [-]

3.7.3.2. « Revisited » Mazars model with regularization (IENDO=9)

In the « revisited » Mazars model, the equivalent strain, now denoted by τ is that proposed by Davenne et al (1989) :

$$\tau = \gamma\sqrt{\langle\varepsilon_1\rangle_+^2 + \langle\varepsilon_2\rangle_+^2 + \langle\varepsilon_3\rangle_+^2}$$

It introduces a coefficient γ, which aims to obtain better results when the material is in biaxial compression. The coefficient γ is defined by the following expression:

$$\gamma = -\frac{\sqrt{\langle\bar{\sigma}_1\rangle_-^2 + \langle\bar{\sigma}_2\rangle_-^2 + \langle\bar{\sigma}_3\rangle_-^2}}{\langle\bar{\sigma}_1\rangle_- + \langle\bar{\sigma}_2\rangle_- + \langle\bar{\sigma}_3\rangle_-}$$

where $\bar{\sigma}_i$ i is the main effective stress in the direction i, and $\langle\bar{\sigma}_i\rangle_-$ its negative part: $\langle\bar{\sigma}_i\rangle_-$ = $\bar{\sigma}_i$i if $\bar{\sigma}_i \leq 0$, $\langle\bar{\sigma}_i\rangle_-$ = 0 otherwise. The value of γ is bounded between 0 and 1, and is calculated only when at least one main stress is negative, that is to say in compression.

On the other hand, the numerical treatment of damage requires a technique of regularization in order to make the solution less dependent on the size of the elements in the mesh. The technique chosen here consists in making the evolution of the damage depend on the size of the finite element. We modify the equation used to calculate the variable d_t according to the proposition of La Borderie (2003):

$$d_t = 1 - \frac{r^0}{r}\exp[-B_t(r - r^0]$$

Parameter B_t depends on the characteristic length l_c, of the fracture energy in mode I, denoted by G_f, and on the tensile strength of the material σ_t :

$$B_t = \frac{l_c\,\sigma_t}{G_f}$$

The characteristic length is chosen as $l_c = \sqrt{S}$, where S is the area of the element in which the integration point is located. The fracture energy G_f is supposed to be a material property – its experimental determination is difficult).

> Parameters for the damage model Mazars revisité avec régularisation (IENDO=9):
>
> - strain defining the initial damage threshold $\varepsilon_D°$ [-]
> - tensile strength σ_t (Pa)
> - fracture energy in mode I, G_f, (Pa/m)
> - A_c, B_c : parameters for the damage law in compression [-]

3.7.3.3. Oliver model (IENDO=7)

To characterize the progressive degradation of the properties of the material, Simo and Ju (1987) propose to use a quantity related to the energy of deformation, noted τ, which expresses the state of deformation in three dimensions:

$$\tau = \sqrt{\varepsilon_{ij}: C^0_{ijkl}: \varepsilon_{kl}} = \sqrt{2\psi^0}$$

The energy τ can also be expressed in terms of principal effective stresses $\bar{\sigma}_i$:

$$\tau = \sqrt{\bar{\sigma}_i \, s^0_{ij} \, \bar{\sigma}_i}$$

This yields in the three-dimensional case:

$$\tau = \frac{1}{E_0}[\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \bar{\sigma}_3^2 - 2\nu(\bar{\sigma}_1\bar{\sigma}_2 + \bar{\sigma}_1\bar{\sigma}_3 + \bar{\sigma}_2\bar{\sigma}_3)]$$

We used the third model proposed in the article by Oliver et al (1990), which modifies the expression of τ in order to better take into account the behavior in compression, by introducing a coefficient γ, a function of the stress tensor:

$$\tau = \gamma \sqrt{\bar{\sigma}_i \, s^0_{ij} \, \bar{\sigma}_i}$$

with

$$\gamma = \left(\theta + \frac{1-\theta}{n}\right) \ ; \ \theta = \frac{\sum_{i=1}^3 \langle \bar{\sigma}_i \rangle}{\sum_{i=1}^3 |\bar{\sigma}_i|} \quad ; \quad n = \frac{\sigma_c}{\sigma_t}$$

where $\bar{\sigma}_i$ i is the main effective stress in the direction i, and $\langle \bar{\sigma}_i \rangle_-$ its negative part: $\langle \bar{\sigma}_i \rangle_-$ = $\bar{\sigma}_i$i if $\bar{\sigma}_i \leq$ 0, $\langle \bar{\sigma}_i \rangle_-$ = 0 otherwise. It follows that γ = 1 in triaxial tension and γ = 1/n in triaxial compression ; σ_c and σ_t are the tensile and compressive strength, respectively.

The damage criterion is given by:

$$F = \tau - r \leq 0$$

where r denotes the actual damage threshold ; its initial value is denoted by r° , given by

r° = $\sqrt{\varepsilon_t \, E_0 \, \varepsilon_t}$

where ε_t is the deformation corresponding to the tensile limit $\sigma_t = E_o \ \varepsilon_t$, E_o designating the modulus of elasticity without damage. The surface of the elastic domain F = 0 is shown in the following figure by way of illustration, σ_c representing the compressive strength.

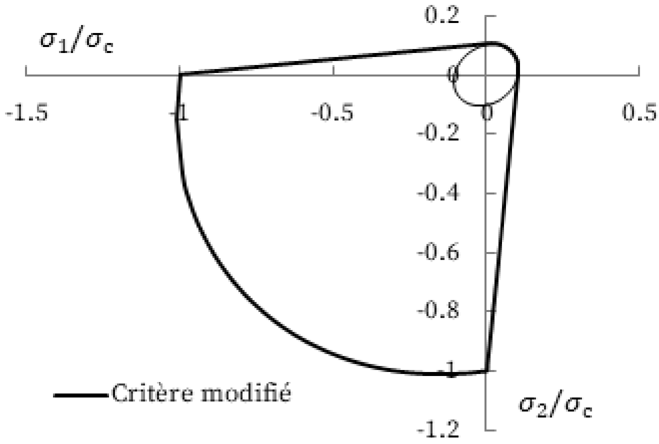


Figure – Limit surface for the Oliver et al (1990) damage model in the principal stresses space with $\sigma_3 = 0$

As regards the evolution of the damage variable, Oliver et al (1990) proposethe following form:

$$d = 1 - \frac{r_0}{r} \exp\{A(1 - \frac{r}{r_0})\}$$

Parameter A takes into account the regularization technique necessary for the numerical treatment of the model. The expression proposed by Oliver et al (1990) is:

$$A = \left(\frac{G_f E_0}{l_c {\sigma_t}^2} - \frac{1}{2}\right)^{-1} \geq 0$$

where G_f is the fracture energy in mode I, considered as a property of the material; the characteristic length is chosen as $l_c = \sqrt{S}$, where S is the area of the element in which the integration point is located.

The type of stress-strain curve for this model is shown in the figure below.

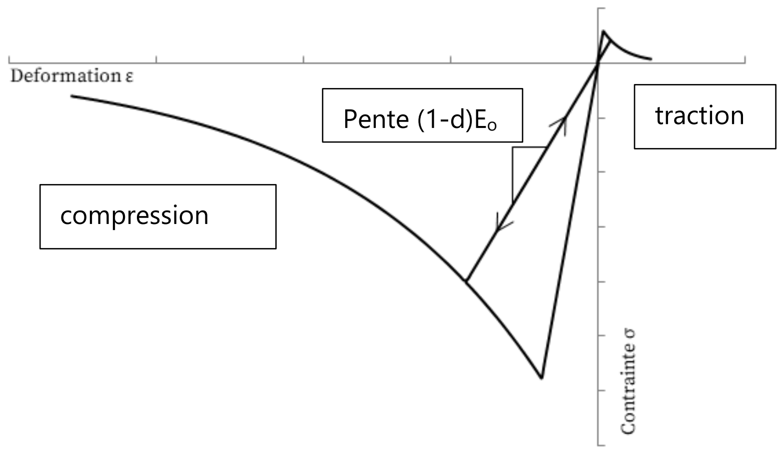


Figure – Uniaxial behaviour for the Oliver model

The Mazars model presented in the previous paragraph and the model proposed by Oliver et al (1990) correctly reproduce the non-linear behavior of concrete in tension. However, Mazars' model probably reproduces with more precision the compressive behaviour of concrete.

Parameters for the damage model Oliver (IENDO=7):

- tensile strength σ_t (Pa)
- fracture energy in mode I, G_f (Pa/m)
- compressive strength σ_c (Pa)

3.7.3.4. Faria model (IENDO=10)

Parameters for the damage model Faria (IENDO=10):

- tensile strength σ_t (Pa)
- compressive strength σ_c (Pa)
- A_c, B_c : parameters for the damage law in compression [-]
- fracture energy in mode I, G_f (Pa/m)

3.7.4. Anisotropic damage models for masonry

Moreno Regan's PhD (led in the context of a CIFRE convention between RATP and IFSTTAR) aimed to develop models of mechanical behavior to represent the masonry vaults of the Paris subway. Elastoplastic calculations show shortcomings: they give an initial phase of relatively stiff deformation, followed by a brutal rupture, while experience indicates that the real behavior of the arch is more flexible. This lack of representativeness of classic models is problematic when it comes to defining displacement thresholds not to be exceeded during work near an old tunnel.

The approach takes into account two aspects of the behavior of masonry:

- the masonry structure leads to representing the vault by a homogenized anisotropic medium. The heterogeneous structure constituted by the blocks and the mortar joints is replaced by an equivalent homogeneous material, according to a particular homogenization technique drawn from the literature: we therefore develop in a first step an orthotropic elastic model, which takes into account the orientation and thickness of joints, fitting of blocks, etc. We introduced an orthotropic linear elasticity (IELAS = 14), then the possibility of calculating the anisotropic elasticity modules from the characteristics of the masonry (IELAS = 15), and finally taking into account the local orientation of masonry joints in an elliptical arch (IELAS = 16/17).

- on the other hand, the literature available on masonry shows that it exhibits damaging behavior, that is to say a loss of stiffness when the deformation it undergoes increases. We have implemented a general algorithm for this type of model, and introduced a set of "classic" isotropic damage models usable for materials other than masonry (for example concrete). The last step was to combine anisotropy and damage to define a model specifically dedicated to masonry. An isotropic damage is introduced for each constituent of the masonry, and to take account of the state of damage of the constituents in the procedure for homogenizing the elastic properties.

The approach can be implemented for different local damage models in the constituents, which explains why there are several damage laws proposed in CESAR for masonry. The approach proposed by Zucchini consists in using a Rankine type criterion. Moreno Regan (2016) proposed two variants, in which local damage is described by the Mazars model or by that of Oliver.

The following models are meant to be associated with the elastic models describing the masonry, IELAS=15 or IELAS=17 in the case of an elliptic vault.

3.7.4.1. Zucchini damage model (IENDO=22)

In the approach proposed by Zucchini (2004), the damage is described by a Rankine type model. In this model, under normal loads, the block and the vertical joint are supposed to be damaged in mode I. The limit surface of damage is given by a Rankine criterion:

$$F = \tau - r \leq \eta_1$$

where $\eta_1 = 10^{-2}$, $\tau = \bar{\sigma}_p$ is the maximum principal effective stress in tension of the block or vertical joint and r is the actual threshold, equal to the initial threshold σ_t if it has never been exceeded, or to the maximum past value of τ otherwise.

The damage law is close to that proposed by Oliver et al (1990) :

$$d = 1 - \frac{\sigma_t}{r}\exp\{A(1 - \frac{r}{\sigma_t})\}$$

Parameter A takes into account the regularization technique necessary for the numerical treatment of the model. The expression proposed by Oliver et al (1990) is:

$$A = \left(\frac{G_f E_0}{l_c {\sigma_t}^2} - \frac{1}{2}\right)^{-1} \geq 0$$

Where G_f is the fracture energy in mode I, considered as a property of the material; the characteristic length is chosen as $l_c = \sqrt{S}$, where S is the area of the element in which the integration point is located.

The horizontal joint (in direction b) is assumed to undergo damage in mode II. The damage criterion is the same, but $\bar{\sigma}_p$ is replaced by $\bar{\tau}$, the effective shear stress, and the tensile strength σ_t by the shear strength σ_s. The evolution law becomes

$$d = 1 - \frac{\sigma_s}{r}\exp\{A_s(1 - \frac{r}{\sigma_s})\}$$

where parameter A_s is calculated by

$$A = \left(\frac{G_{fs} {G_m}^{\circ}}{l_c {\sigma_s}^2} - \frac{1}{2}\right)^{-1} \geq 0$$

where l_c is the characteristic length, G_{fs} the fracture energy in mode II and $G_m{}^{\circ}$ the shear modulus (of the mortar) prior to any damage.

Parameters for the Zucchini damage model (IENDO=22):

- Tensile strength of the blocks σ_{tB} (Pa)
- Fracture energy in mode I of the blocks G_{fB} (Pa/m)
- Tensile strength of the mortar σ_{tm} (Pa)
- Fracture energy in mode I of the mortar G_{fm} (Pa/m)
- Shear strength of the blocks σ_{sB} (Pa)
- Fracture energy in mode II of the blocks G_{fsB} (Pa/m)
- Tensile strength of the mortar σ_{sm} (Pa)
- Fracture energy in mode II of the mortar G_{fsm} (Pa/m)
- Compressive strength of the blocks σ_{cB} (Pa)

- Compressive strength of the mortar σ_{cm} (Pa)

3.7.4.2. Zucchini +Mazars damage model (IENDO=24)

This is a variant in which the damage to the constituents is described by the revisited Mazars model with regularization. The corresponding parameters are given for the two constituents (blocks and mortar).

Parameters for the Zucchini +Mazars damage model (IENDO=24):

- Tensile strength of the blocks σ_{tB} (Pa)
- Fracture energy in mode I of the blocks G_{fB} (Pa/m)
- Tensile strength of the mortar σ_{tm} (Pa)
- Fracture energy in mode I of the mortar G_{fm} (Pa/m)
- valeur du seuil d'endommagement initial of the blocks ε_{DB} (-)
- valeur du seuil d'endommagement initial of the mortar ε_{Dm} (-)
- A_{cB}, B_{cB} : parameters for the damage law in compression of the blocks [-]
- A_{cm}, B_{cm}: parameters for the damage law in compression of the mortar [-]

3.7.4.3. Zucchini+Oliver damage model (IENDO=26)

In this variant, the damage in the constituents is described by the Oliver model. The corresponding parameters are given for the two constituents (blocks and mortar).

Parameters for the Zucchini+Oliver damage model (IENDO=26):

- Tensile strength of the blocks σ_{tB} (Pa)
- Fracture energy in mode I of the blocks G_{fB} (Pa/m)
- Tensile strength of the mortar σ_{tm} (Pa)
- Fracture energy in mode I of the mortar G_{fm} (Pa/m)
- Compressive strength of the blocks σ_{cB} (Pa)
- Compressive strength of the mortar σ_{cm} (Pa)

4. Multiphase modelling of reinforced materials

4.1. Principle of the proposed model

This section deals with models that are available in the solver, but for which the preparation of the data is not possible using the graphical interface CLEO.

The approach is only useable in mechanics for static computations.

Only one set of reinforcement inclusions can be taken into account.

4.2. Modèles de comportement pour la phase matrice

All the material models for the standard bulk elements (families 01 and 02) either classical or user defined, can be used with the multiphase elements (families 431/432).

4.3. Geometrical arrangement of the reinforcement inclusions (indicator IGEOM)

The same geometries can be taken into account as for IMOD=43 or with the keyword RENF in the user defined models. Please refer to 3.2.20 and 3.3.5.2.

4.3.1. Homogeneous reinforcement (IGEOM=1)

4.3.2. Radial reinforcement (IGEOM=2)

4.3.3. Cylindrically diverging reinforcement (IGEOM=3)

4.3.4. Spherically diverging reinforcement (IGEOM=4)

4.4. Mechanical properties for the reinforcement (indicator ICOMP)

The same options are available as in IMOD=43 or with the keywork RENF in the user defined models. Please refer to 3.2.20 and **Erreur ! Source du renvoi introuvable.**.

4.4.1. Linear elasticity (ICOMP=1)

4.4.2. Linear elasticity + perfect plasticity (ICOMP=2)

4.4.3. Linear elasticity + brittle failure (ICOMP=3)

4.4.4. Linear elasticity + perfect plasticity + brittle failure (ICOMP=4)

4.5. Models for the mechanical interaction between ground and inclusions (indicator ICINT)

The improvement of the approach with respect to IMOD=43 or RENF in the user defined models lies in the fact that the user can explicitly introduce a model for the mechanical interaction between the ground and the inclusions.

4.5.1. Linear interaction (ICINT=1)

In the model, I is a volume density of interaction force (N/m^3) ; ξ is the relative displacement between the ground and the reinforcement (along the direction of the inclusions).

Formulation : $I = C_{int}\ \xi$

The volumic interaction coefficient is here in N/m^4 (or in Pa/m^2)

4.5.2. Linear elastic perfectly plastic interaction (ICINT=4)

Formulation : $I = C_{int}(\xi-\xi^p)$

The volumic interaction coefficient is here in N/m^4 (or in Pa/m^2)

The maximum value of the interaction force is Imax (in N/m^3)

4.5.3. Bilinear elastic perfectly plastic interaction (ICINT=14)

-

4.5.4. Parabolic elasticity with perfect plasticity interaction (ICINT=15)

-

5. Bulk elements in dynamics

5.1. Linear computations (module DYNI)

The DYNI computation module only allows to use linear models.

5.1.1. Classical models (IMOD=1 and 2)

The models IMOD=1 and IMOD=2, available for static simulations, are also available in dynamics with DYNI . Please refer to 0 and 0.

5.1.2. Damping by group (IMOD=66)

The DYNI module takes into account a Rayleigh type of damping: the damping matrix is a linear combination of the stiffness matrix and of the mass matrix. By default, the coefficients are the same for all groups of elements in the mesh, and given in the data of the module (option AMO).

In some cases, it is necessary to consider different dampings in different ground layers. To input specific coefficients in each group, one can use the IMOD=66 model ().

5.1.3. User defined models

All the linear elastic models described in the user defined models (3.3.3) can be used, including transversely isotropic or orthotropic models, or heterogeneous but otherwise linear models : IELAS=0, 1, 8, 2, 9, 11, 14, 15, 16, 17.

The use of non linear model is not possible with DYNI.

5.2. Non linear computations (module MCCI)

To be completed.

6. Bulk elements for coupled hydro-mechanical or thermo-poro-mechanical analyses

6.1. Hydro-mechanical coupling

In principle, all the constitutive models that can be used for the bult elements in statics are also available for computations using the module CSNL : the classical models and the user defined models.

For classical models, the datafile contains the mechanical properties (for the same value of IMOD as in statics) followed by the hydraulic properties.

For the user defined models, a specific keyword analogous to RHO, ELAS, CRT, POT or RENF is introduced, the keyword 'HYDR' followed by the hydraulic properties (see the reference manual of the solver).

6.2. Thermo-poro-mechanical coupling

Contrary to CSNL, module MPNL does not make it possible to use all the models available in uncoupled static analyses. The list of models compatible with MPNL correspond to IMOD=1, 2, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20.

7. Interface elements

7.1. “Contact” elements and joint elements

CESAR proposes two families of finite elements to deal with interfaces in a mechanical analysis: the so-called “contact elements” of family 06 and the joint elements (famiy 16). The contact elements need to be used with the computation module TCNL, where the joints can be used with the module MCNL.

7.2. Joint elements

7.2.1. General remarks about contact elements

To model the mechanical behaviour of interfaces between deformable solids, the solver CESAR was equipped with a specific computation module called TCNL, in which interfaces were dealt with by means of a specific family of elements : the family 06 of the so-called « contact elements » (or in some cases friction/opening elements). Starting from version 2020 of CESAR-LCPC, another possibility is now available: a new family of elements called « joint elements » (after Goodman (1968)) to avoid confusion with the previous one. The interest of the joint elements family lies in the fact that it works with the computation module MCNL. It is described in the following chapter (7.3-Joint elements).

The material non-linearities in the bulk elements in TCNL is the same as in MCNL with the « initial stiffness » solution algorithm (IMET=1 in the data of the computation module MCNL). This section focuses on the treatment of the interface.

Consider two solids Ω_1 and Ω_2 potentially in contact in a zone S_C (see figure). Initially, the contact surface is $S_0 \subseteq S_C$. The principle of the TCNL algorithm consists in applying the load in an incremental way in order to monitor the evolution of the actual contact surface S ($\subseteq S_C$). For each load increment, an iterative procedure is used to check simultaneously the equilibrium equations and the contact criteria.

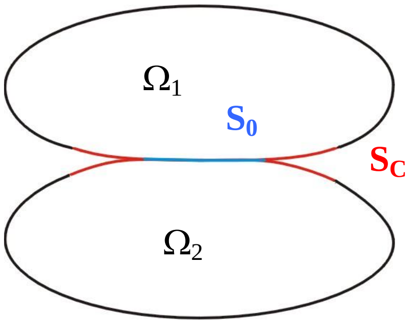


Figure 11 – Solids in contact

The contact is said to be rigid if the solids can enter in contact and deform each other but not interpenetrate. The distance d between a point of domain Ω_1 to domain Ω_2 after application of the load is thus positive or zero (see the following figure). The condition $d \geq 0$ is a non-interpenetration condition.

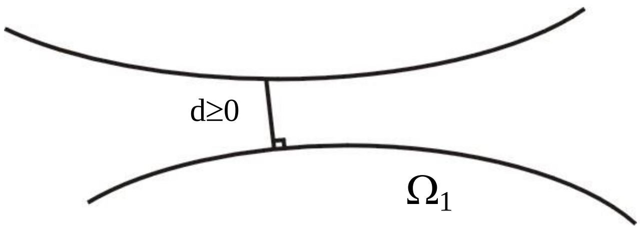


Figure 12 – Non penetration

Friction criterion

When the solids Ω_1 and Ω_2 are in contact, the relative tangential displacement between the solids is either zero (perfect bonding) or non zero (sliding contact). The nature of the forces acting at the interface between solids is different depending on the contact nature. We denote by σ^1 and σ^2 the normal and tangential stresses at the contact point. The explanations below are described in a bidimensional situation, but the three-dimensional case is treated in a similar way. In the local axes (n_1,t_1) of the following figure Ω_1 :

$$\sigma_1 = \begin{pmatrix} \sigma_n^1 \\ \\ \tau_1 \end{pmatrix} \qquad (1)$$

In a simliar way, the stress in the second body is σ^2 in (n_2,t_2), with n_2=-n_1 et t_2=-t_1. Since $\sigma^1+\sigma^2=0$, we only refer to the stresses in Ω_1, ommitting the indices attached to the solid.

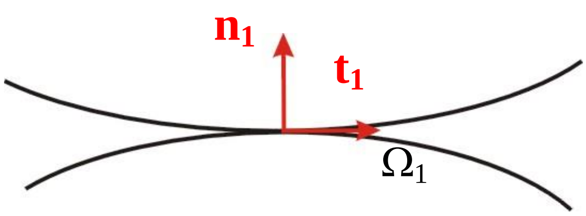


Figure 13 - Friction

The Coulomb friction law is described by the yield function:

$$f = |\tau| - c + \sigma_n \tan \phi \qquad (2)$$

where c is the cohesion and ϕ the friction angle of the interface. These quantities are characteristic of the nature of the contact between the solids under consideration. As show in the following figure, Coulomb's law gathers three situations. The contact is adherent if f<0, slinding if f=0, and f>0 is excluded. When sliding occurs, tne relative displacement ε_C between solids at the contact point is given by:

$$\varepsilon_C = \lambda \cdot \frac{\partial g}{\partial \sigma} \qquad (3)$$

where λ is a plastic multiplier to be determined, $g = |\tau| - c + \sigma_n \tan \psi$ is the plastic potential, and ψ the dilatancy angle. When $\psi \neq \phi$, the friction model is said to be non-associated. The traction resistance is expressed by the condition $\sigma_n \leq R_t$. The choice of R_t is discussed hereafter.

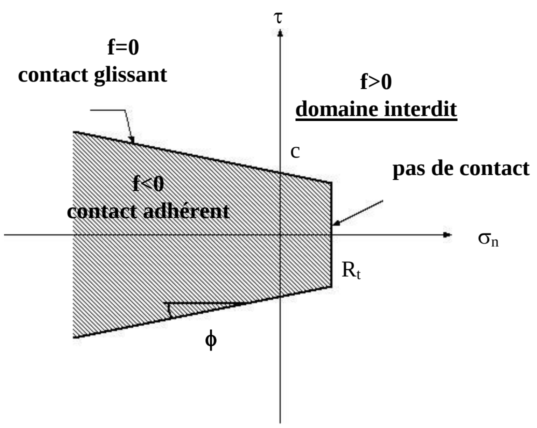


Figure 14 – Mohr-Coulomb model with cohesion and traction resistance

In the elements of contact of CESAR, the modelling is based on a regularization method by penalization, meanin that the contact law is only verified approximately. The procedure can be interpreted as follows:

- in the contact area, it is tolerated that the solids interpenetrate slightly. In other words, negative values of d are allowed, provided that they remain small with respect to the solids diameters.

- the relative tangential displacement may be non zero even in the adherent case (f<0).

The technique adds up to considering that in relative displacement between the bodies there are a plastic (irreversible) part (corresponding to friction) and an elastic part (due to the penalization).

7.2.2. Interface models for contact elements

The contact elements in CESAR are:

- in 2D or axisymmetrical condition: 6-node quadrilateral element.

- in 3D : 16 nodes hexhaedra or 12 nodes pentahedra.

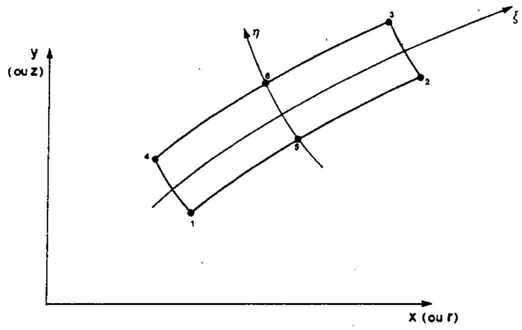


Figure 15 – 6 node quadrilateral element

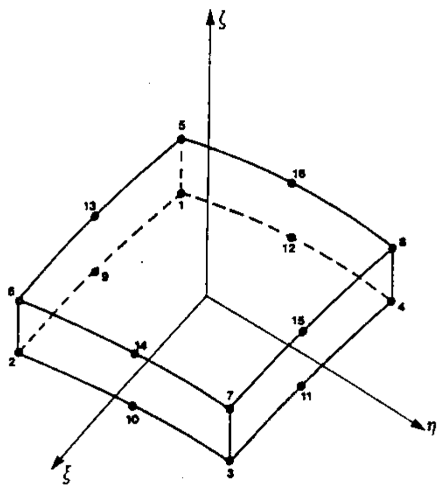


Figure 16 - 16 node hexahedron

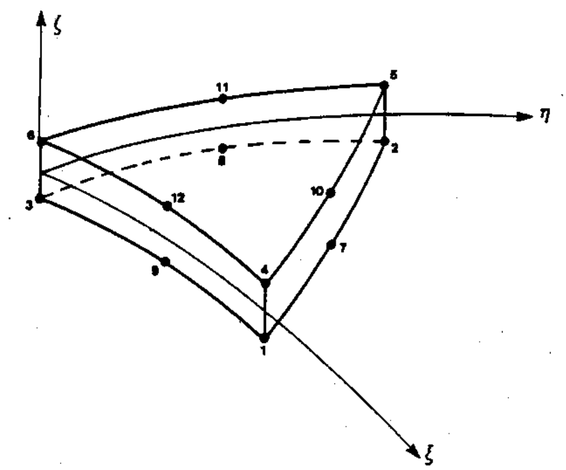


Figure 17 – 12 nodes pentahedron

All the elements of this family have two « faces » and the code computes the stresses at the interface using the relative displacement between the two homologous nodes on both faces. To take these elements into account in the finite element computation, they are given a stiffness matrix, computed with a fictitious coefficient provided by the user ; it is recommended to take it equal to the smallest of the Young's moduli of the two bodies in contact.

7.2.2.1. Bonding (IMOD=1)

In this case, the interface elements actually ensure the continuity of the displacements between the two faces of the element. There is no limit to the shear stress that can take place at the interface.

7.2.2.2. Mohr-Coulomb friction (IMOD=2)

The Mohr Coulomb model in the interface corresponds to the following strength condition:

$$|\tau| < c - \sigma_n \tan \phi$$

where τ is the shear stress, σ_n the normal stress, c and ϕ the cohesion and the friction angle of the interface.

7.2.2.3. Perfect sliding (IMOD=3)

In this case, the contact elements only ensure the continuity of the normal displacement. There are no shear stress at the interface.

7.2.2.4. Mohr-Coulomb – alternative algorithm (IMOD=4)

This is another numerical implementation of the same theoretical model as IMOD=2. Both implementations are supposedly equivalent in 2D. In 3D, the numerical treatment with IMOD=4 may provide a solution in situations where the standard model does not perform well.

7.2.2.5. Use of interface elements for diffusion computations

It can be useful to perform mechanical and hydraulic (or thermal) simulations with the same mesh. Strating from version 2020 of CESAR, it is possible to convert the mechanical interface elements (contact elements) into elements with a high conductivity or permeability – on into elements providing perfect thermal insulation (or perfect impermeability). The technique consists in replacing the elements of family 06 by elements of family 28. This conversion is performed automatically by the graphical interface CLEO (or can be performed manually by editing the datafile).

7.3. Joint elements

7.3.1. General remarks about joint elements

The joint elements are meant to deal with the same type of problems as the contact elements of the previous chapter but with the computation module MCNL. The numerical treatment is simpler than that of the contact elements: they are treated like the other elastoplastic elements. Basically, they rely on the approach proposed by Goodman (1968).

As for the contact elements, the relative displacement between the solids in contact is the sum of a plastic (irreversible) part corresponding to sliding, and an elastic part (reversible) that is bound to remain small as long as the shear resistance of the interface is not reached.

7.3.2. Interface models for joint elements

The joint elements in CESAR are:

- in plane strain or in axisymmetric condition : quadrilateral elements with 4 or 6 nodes.

- in 3D : hexahedra with 8 or 16 nodes or pentahedra with 6 or 12 nodes.

All the elements of this family have two « faces » and the code computes the stresses at the interface using the relative displacement between the two homologous nodes on both faces. To take these elements into account in the finite element computation, they are given a stiffness matrix, computed with a fictitious coefficient provided by the user ; it is recommended to take it equal to 100 times the smallest of the Young's moduli of the two bodies in contact.

Bonding (IMOD=1)

In this case, the joint elements actually ensure the continuity of the displacements between the two faces of the element. There is no limit to the shear stress that can take place at the interface.

Mohr-Coulomb friction (IMOD=2)

The Mohr Coulomb model in the interface corresponds to the following strength condition:

$$|\tau| < c - \sigma_n \tan \phi$$

where τ is the shear stress, σ_n the normal stress, c and ϕ the cohesion and the friction angle of the interface.

Perfect sliding (IMOD=3)

In this case, the joint elements only ensure the continuity of the normal displacement. There are no shear stress at the interface.

7.3.2.1. Use of joint elements in diffusion computations

It can be useful to perform mechanical and hydraulic (or thermal) simulations with the same mesh. Strating from version 2020 of CESAR, it is possible to convert the mechanical interface elements (joint elements) into elements with a high conductivity or permeability – on into elements providing perfect thermal insulation (or perfect impermeability). The technique consists in replacing the elements of family 16 by elements of family 28. This conversion is performed automatically by the graphical interface CLEO (or can be performed manually by editing the datafile).

8. Bar elements

8.1. Introduction

Bar elements are 1-dimensional elements, in which no effort is taken into account other than the axial force N. It is related to the longitudinal strain ε, defined as the change in length of the bar δ divided by its initial length. In the elastic linear case, the constitutive law is expressed by:

$$N = E\,S\,\delta/L = E\,S\,\varepsilon$$

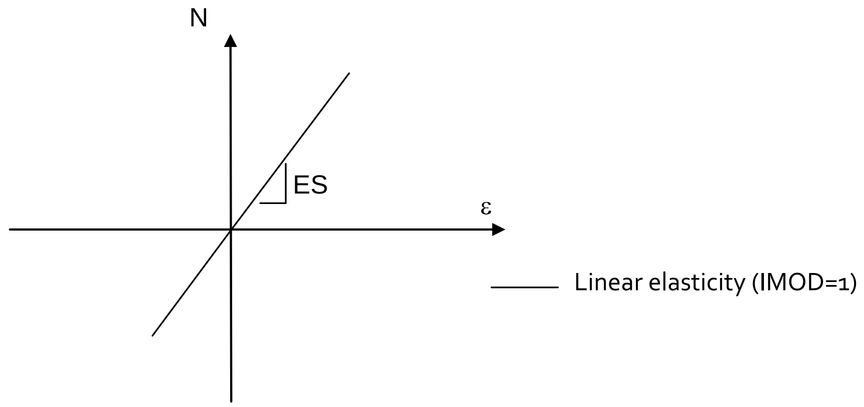
where E is the Young' modulus of the material constituting the bar and S its cross sectional area (supposed to be constant over the element).

In the case of an elastic-perfectly plastic behaviour, there is a linear relation between N and ε as long as the axial force N remains less than a given value F_{max}, the proportionality coefficient being again equal to the product of the Young' modulus of the material constituting the bar and its cross sectional area. When the axial forces reaches the maximum force F_{max}, the bar can stretch under constant axial force: the strain in the bar is the of two contributions, one reversible, denoted by ε^e, and equal to $N_o L/ES$, the other irreversible, denoted by ε^p. The value of F_{max} is the product of the tensile strength of the material and of the cross sectional area. In some cases, it could be useful to consider a limit value in compression different from the value in tension.

Beyond these two simple models, CESAR offers seveal other nonlinear models for bar elements. They are available in 2D or in 3D, for bar elements with 2 or 3 nodes.

8.2. Constitutive models

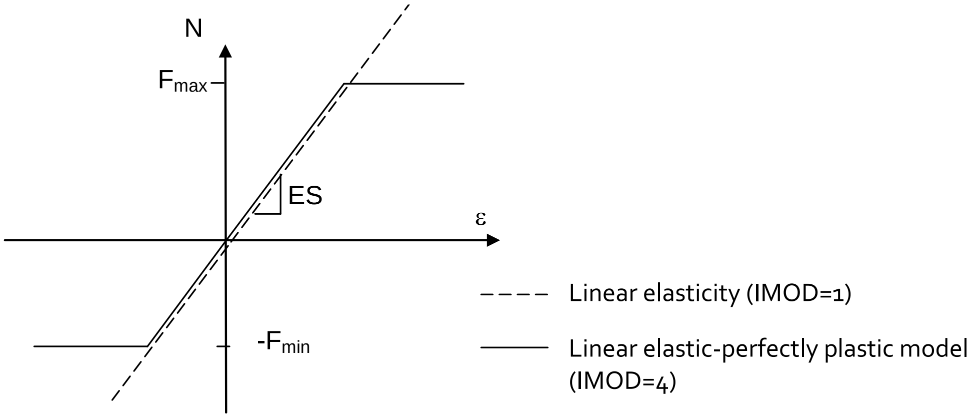
8.2.1. Linear elasticity (IMOD=1)



Parameters for the linear elastic model for bar elements (IMOD=1) :

- Young's modulus (`YOUNG`) [Pa] ;
- cross-sectional area (`S`) [m^2].

8.2.2. Linear elastic-perfectly plastic model (IMOD=4)

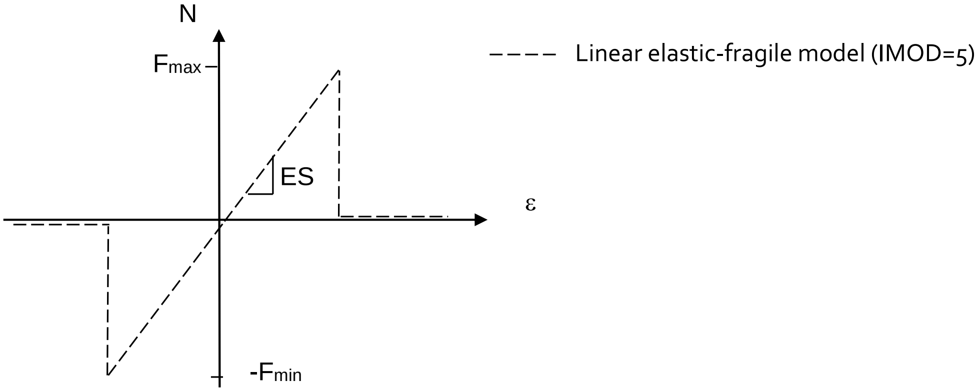


Parameters for the linear elastic-perfectly plastic model for bar elements (IMOD=4) :

- Young's modulus (`YOUNG`) [Pa] ;
- Maximum value of the axial force in tension (`FMAX`) [N] ;
- Maximum value of the axial force in compression (`FMIN`) [N] ;
- cross-sectional area (`S`) [m^2].

8.2.3. Linear elastic- brittle model (IMOD=5)

CESAR offers the possibility to model a fragile behaviour for bars : when the elastic limit is reached, the axial strain in the bar becomes zero (figure). The nodal forces that the bar sustained are redistributed on the other elements of the mesh having nodes in common with the broken bar.

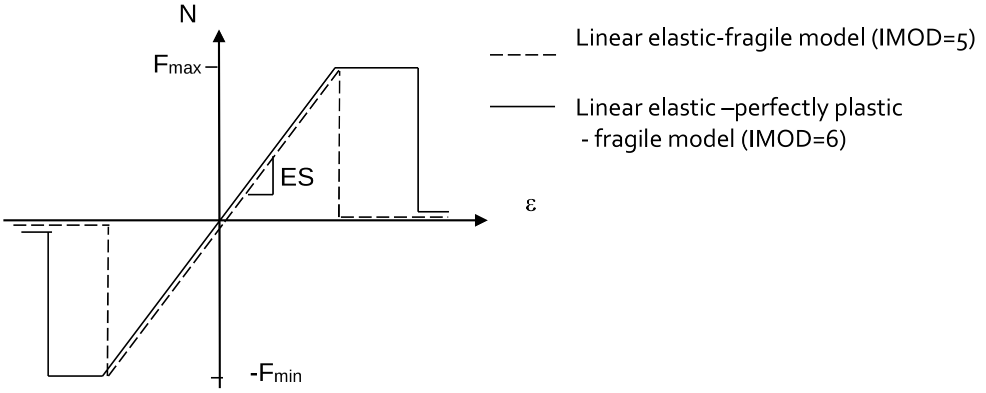


Parameters for the linear elastic- fragile model for bar elements (IMOD=5) :

- Young's modulus (`YOUNG`) [Pa] ;
- Maximum value of the axial force in tension (`FMAX`) [N] ;
- Maximum value of the axial force in compression (`FMIN`) [N] ;
- cross-sectional area (`S`) [m^2].

8.2.4. Linear elastic-perfectly plastic- brittle model (IMOD=6)

It is also possible to account for a linear elastic-perfectly plastic model until a certain strain is reached, beyond which the bar breaks (see figure).



Parameters for the linear elastic-perfectly plastic-fragile model for bar elements (IMOD=6) :

- Young's modulus (`YOUNG`) [Pa] ;
- Maximum value of the axial force in tension (`FMAX`) [N] ;
- Maximum value of the axial force in compression (`FMIN`) [N] ;
- Maximum strain at failure (tension) (EPSMAX) [-] ;
- Maximum strain at failure (compression) (EPSMIN) [-] ;
- cross-sectional area (`S`) [m^2].

8.2.5. Linear elastic-perfectly plastic with loss of stiffness model (IMOD=7)

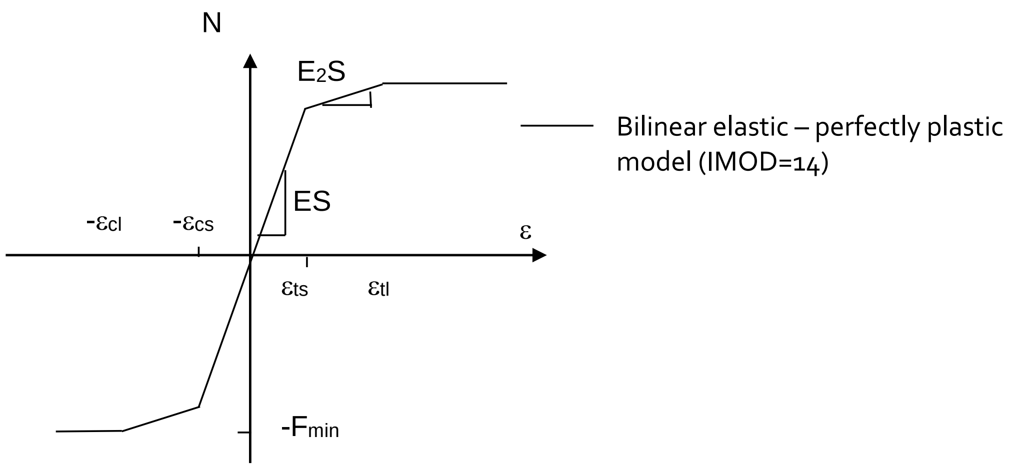
This model has been proposed to take into account a decrease in the Young's moduli of the bar between two successive computation steps in a phased analysis. The loss of stiffness results in a decrease in the axial force in the bar, which is partly redistributed on the adjacent elements. The user needs to input the actual modulus and its valus at the previous computation step.

Parameters for the linear elastic-perfectly plastic with loss of stiffness model for bar elements (IMOD=7) :

- Young's modulus (`YOUNG`) [Pa] ;
- Maximum value of the axial force in tension (`FMAX`) [N] ;
- Maximum value of the axial force in compression (`FMIN`) [N] ;
- Young's modulus at the previous computation step (EREF) [Pa] ;
- cross-sectional area (`S`) [m^2].

8.2.6. Bilinear elastic perfectly plastic model (IMOD=14)

This model proposes a bilinear elastic law before the plastic flow regime is reached (see figure).



Parameters for the bilinear elastic perfectly plastic model for bar elements (IMOD=14):

- initialYoung's modulus E (YOUNG) [Pa] ;
- secondary Young's modulus E2 (YOUNG2) [Pa] ;
- axial strain at the transition between the moduli, in extension ε_{ts}(EPSTS) [-] ;
- axial strain for the onset of plasticity , in extension ε_{tl}(EPSTL) [-] ;
- axial strain at the transition between the moduli, in compressionε_{cs}(EPSCS) [-] ;
- axial strain for the onset of plasticity , in compression ε_{cl}(EPSCL) [-] ;
- cross-sectional area (S) [m^2].

8.2.7. Non available models

The set of models proposed above could be extended to include other phenomena such as a hardening plasticity, or a loss of strength of the bar (caused par corrosion for instance).

9. Bar elements with friction interaction

9.1. Bar with friction interaction elements

The usual bar elements, used in combination with bulk elements to represent reinforced structures (mechanically stabilized earth walls, nailed embankments, etc.), do not limit the effort transmitted to the bars by the ground: the interaction mechanical between the bar and the material which surrounds it comes down to a condition of perfect adhesion. In other words, the interface between the materials cannot fail.

This type of model does not allow a detailed description of the mechanical interaction between the bar and the surrounding material. One solution consists in introducing a richer kinematic description, which allows a relative displacement δ between the bar and the ground. One can then connect the (linear density of) force transmitted to the bar, noted I, with this relative displacement, and compel it to respect a criterion of resistance, to take into account a limit on the friction that can be mobilized at the interface.

Different models are available, which are formally analogous to the models introduced for the bars themselves, but relate to their interaction with the solid elements with which they have common nodes.

NB: these elements belong to a specific family of elements (family 435, codes KRB2 / KRB3 in 2D, KRT2 / KRT3 in 3D, the last digit corresponding to the number of nodes of the element).

9.2. General features

The input data are twofold :

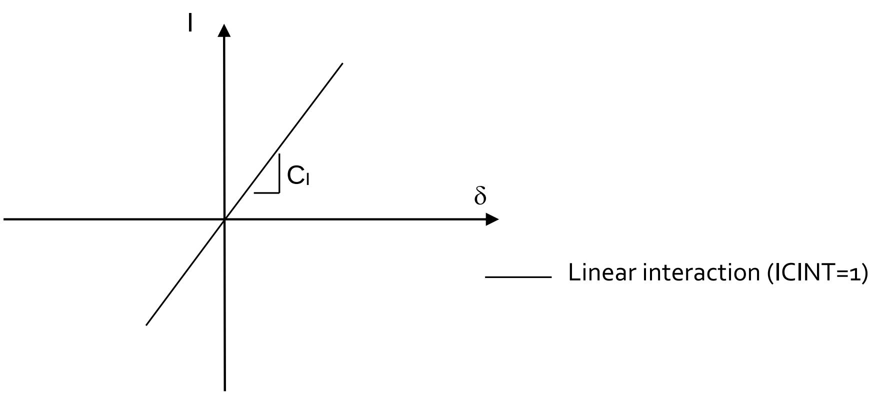
- on a first line, the parameters describing the behavior of the bar are given;
- then on a second line, the parameters associated with the model describing the mechanical interaction between the bar and the surrounding material.

The data structure is therefore similar to that of the user defined models for solid elements, except that there is no explicit keyword in the data set.

9.3. Interaction models

9.3.1. Linear elastic interaction (ICINT=1)

The simplest model corresponds to a linear relationship between the interaction force (in N / m) and the relative displacement δ (in m). The interaction coefficient Ci (such that $I = C_i\,\delta$) has the dimension of a stress.

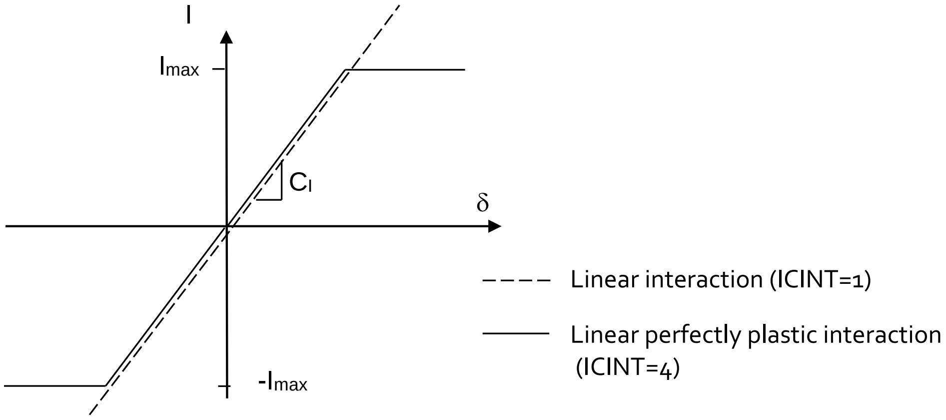


Parameters for the linear elastic bar-ground interaction model (ICINT=1):

- interaction coefficient (`Ci`) [N/m^2] .

9.3.2. Linear elastic –perfectly plastic interaction (ICINT=4)

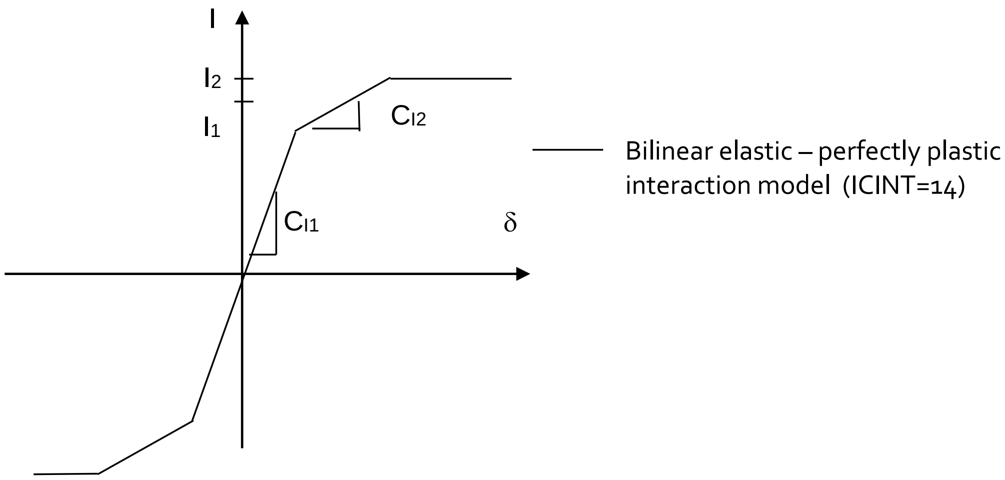
This model limits the force that can be transmitted to the bar per unit of length to the I_{max} value. Beyond, the bar slides in the surrounding material.



Parameters for the linear elastic –perfectly plastic bar-ground interaction model (ICINT=4):

- interaction coefficient (`Ci`) [N/m^2] ;
- Maximum value of the interaction force (`IMAX`) [N/m] .

9.3.3. Bilinear elastic –perfectly plastic interaction model (ICINT=14)

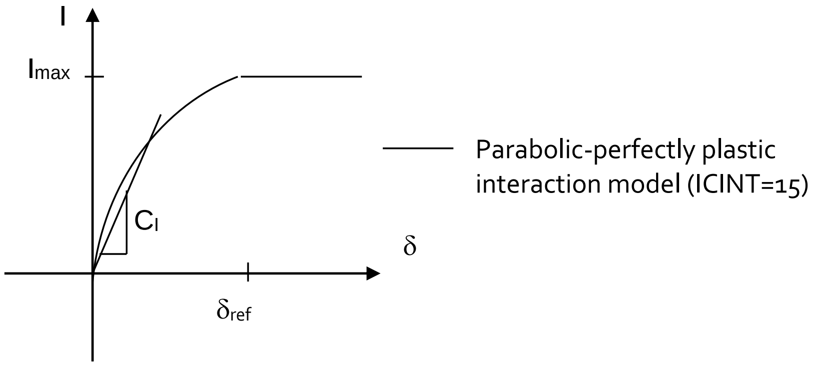


Parameters for the bilinear elastic –perfectly plastic bar-ground interaction model (ICINT=14):

- initial interaction coefficient C_{I1} (`CI1`) [Pa] ;

- secondary interaction coefficient C_{I2} (CI2) [Pa] ;
- Value of the interaction force at the transition between the two values of the interaction coefficient I1(I1) [N/m] ;
- Maximum value of the interaction force (I2) [N/m].

9.3.4. Parabolic-perfectly plastic interaction model (ICINT=15)



$$I = \mathrm{Min}\left[\, C_I\delta,\ I_{max}\,(\delta/\delta_{ref})^{1/2},\ \ I_{max} \right]$$

Parameters for the Parabolic-perfectly plastic bar-ground interaction model (ICINT=15) :

- initial interaction coefficient C_I (CI) [Pa] ;
- Relative displacement δref for which the maximum interaction force is reached (DREF) [m] ;
- Maximum value of the interaction force (IMAX) [N/m]

9.3.5. Bilinear elastic–plastic interaction “Newtun” models (ICINT=16 / 17)

The FUI Newtun project (2012-2016), led by Solétanche Bachy, gave rise to a collaboration with Armines, which carried out pull-out tests on bolts, using an experimental device specifically adapted to the characteristics of the materials tested. This work led to the identification of models characterizing the interaction. Two of these models are presented here and have been integrated into CESAR.

The two proposed models differ by the description of the interaction (by a linear force or by a limiting shear stress) and by the parameters used to describe the interaction curve, but remain very similar. Finally, it should be noted that the experimental results obtained by Armines are much richer than what has been presented here; in particular, they gave rise to the development of a third interface model, which proposes a more progressive law of variation for the linear force of interaction and which describes a resistance peak of the soil-bolt interaction. We could consider expanding the range of models available in CESAR to integrate these results.

Interpretation of experimental results

The preparation of the tests carried out by Armines involves several stages: a sample of an artificial material is formed in a cylindrical tank, the composition and mechanical characteristics of which are controlled, drilling is carried out in the sample according to processes which reproduce drilling bolts on a real tunnel site, a fiberglass inclusion is placed in the borehole which is sealed to the sample using a resin of the type used on site. The test involves exerting an axial force to tear off the bolt from the sample. The rupture occurs at the interface between the resin and the soil sample. The test gives a curve giving the force F applied to the head of the bolt as a function of its displacement δ.

After verifying that the results were reproducible, the test campaign made it possible to study the influence of various parameters, such as the sealing length L, and the confining stress around the bolt. We propose to model the experimental curves by trilinear curves.

The measured force integrates, over the length of the bolt, the shear forces at the interface between the seal grout and the ground. The results can be directly exploited, assuming that the deformability of the fiberglass bolt is very low compared to that of the ground: the pullout measured is then practically equal to the relative displacement between the ground and the bolt over all its length; it is also assumed that the force is distributed more or less uniformly over the length of the bolt. We therefore define the linear force of interaction between the ground and the bolt, denoted I, as the ratio F / L.

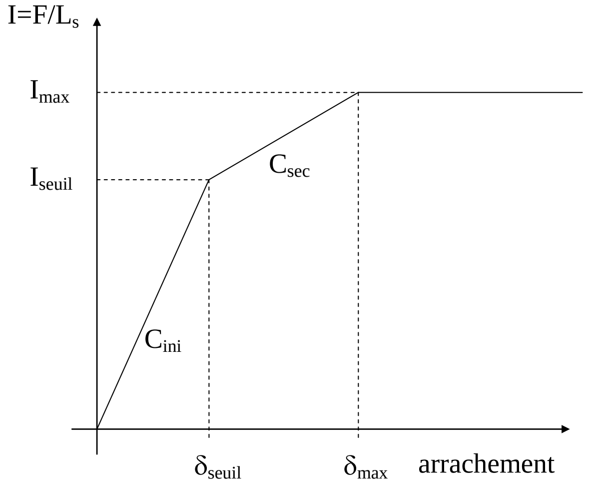
Within this set of assumption, the trilinear model is defined by 4 parameters:

- the maximum value of the interaction force I, denoted Imax, represents the maximum force that the bolt can transmit to the ground per unit of length;

- the Iseuil value of the interaction force I corresponding to the first slope change;

- the two corresponding values of the relative displacement, which one will note δmax and δseuil

(We can also choose as parameters Imax and Iseuil, and the two slopes of the curve I-δ, the initial slope Cini and the secondary slope Csec.



Armines's experimental setup was used to control the confining stress around the bolt. For tunnel calculations, we can make two assumptions about the state of stress that prevails in the soil around the bolts after their installation:

- the confining stress is directly controlled by the grout injection pressure;

- the confining stress is linked to the depth of the bolt, the weight of the earth, and the coefficient Ko:

$$pc = \gamma \, z \, (1 + 2Ko) / 3$$

In this second case, there is a variation in the mechanical characteristics of the ground-bolt interaction between the top and bottom of the front (in particular for a shallow tunnel).

To cover the two situations, we take into account a value of pc given by:

pc = A z + B

Experimental results show that the Iseuil / Imax and Dseuil / Dmax ratios vary little from one test to another. We propose to synthesize the results by taking:

$p_c = A\,z + B$

$I_{max} = c_1\,p_c + c_2$ $\delta_{max} = c_3\,p_c + c_4$

$I_{seuil} / I_{max} = c_5$ $\delta_{seuil} / \delta_{max} = c_6$

$C_{ini} = I_{seuil} / \delta_{seuil}$ $C_{sec} = (I_{max} - I_{seuil}) / (\delta_{max} - \delta_{seuil})$

Parameters for the bilinear-perfectly plastic interaction model Newtun-I (ICINT=16) :

- A [Pa/m], B [Pa] : parameters for the variation of pc with z ;
- c1, c2, c3, c4, c5, c6

We can also interpret the results differently, in a way that gives more weight to the change in value of the slope of the curves. The parameters Fmax / Fseuil or Imax / Iseuil are chosen as parameters. On the other hand, we characterize the resistance of the interface not in terms of linear force of interaction, but of shear stress at the interface; this makes it possible to transpose the results of a test for a given borehole diameter to a bolt in a larger or smaller borehole, the linear interaction force and the shear stress τ being linked by:

$\pi\, d\, L\, \tau = I$

where d denotes the diameter of the sealing / soil interface (assumed to be cylindrical).

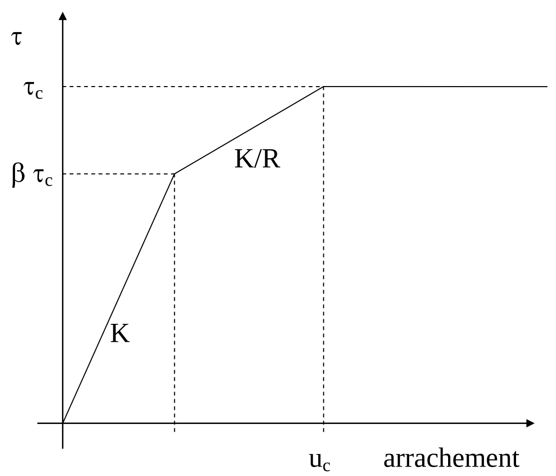
The maximum value τc of the shear stress along a bolt is then given, as a function of the confinement stress pc by: $\tau_c = \alpha p_c + \tau_o$

The relative displacement for which the interface fails is given by:

$u_c = u_{ref}\,(\tau_c / p_c) - u_o$

The initial value of the stiffness of the interface is noted K. When the shear stress exceeds a threshold noted $\tau_t = \beta\,\tau_c$, the stiffness is divided by R.

The reference value of R is 5, and that of β is 0.75.



The displacement value which corresponds to the transition between the two stiffness values is equal to ut = $\beta\tau_c$ /K.

For [ut, uc], we have:

$\tau = \tau t + K/R\ (u - ut)$

The condition τ (u=uc) = τc gives :

$\beta\tau c + K/R\ (uc - \beta\tau c /K) = \tau c$

and allows to set the value of the initial stiffness

$$K = \frac{\tau c\ [R-\beta\ (R-1)]}{uc}$$

and finally Cini is given by: Cini = π d K

In summary, the interaction model ICINT = 17 introduced in CESAR is described by the following equations:

Input parameters: A, B, d, α, τo, uref, uo, β, R

pc = A z + B ; $\tau c = \alpha\ pc + \tau o$

uc= uref (τc /pc) – uo ; $K = \frac{\tau c\ [R-\beta\ (R-1)]}{uc}$

Cini = π d K ; ; Csec = Cini/R ;

Imax = π d τc ; ; Iseuil = β Imax

It is suggested to maintain β and R at their reference values.

Parameters for the bilinear-perfectly plastic interaction model Newtun-II (ICINT=17) :

- [Pa/m], B [Pa] : parameters for the variation of pc with z ;
- d diameter of the inclusion
- α, τo : parameters for the variation of the maximum shear stress with pc
- uref, uo : parameters for the sliding at which the interface fails
- β ratio of the shear stress for which the stiffness changes, and the maximum shear stress
- R ratio of the initial stiffness to the secondary stiffness

9.3.6. Unavailable features

There are no models with a complete failure of the interface (the interaction force being reduced to zero), and to negative hardening.

10. Beam elements

CESAR proposes two types of beam elements : classical beam elements and multifibre beam elements (only available in 3D). For sake of simplicity, there is only only family, and the choice between the types of elements is determined by the value of IMOD.

10.1. Classical beam elements

10.1.1. Linear elasticity (IMOD=1, 12, 13, 14, 15, 16, 101 , 113, 114, 115 ou 116)

For the classical beam elements in CESAR, the basic constitutive model is linear elasticity IMOD=1.

Several variants (IMOD= 12, 13, 14, 15, 16, 101 , 113, 114, 115 ou 116) have been implemented to test the influence of the numerical integration scheme, or the choice to include or nor some terms in the elastic energy. The variants lead to more or less stiff responses. The parameters are the same as for the basic implementation.

Parameters for linear elasticity for 2D classical beam elements

Mechanical parameters :

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]

Geometrical parameters :

- Cross-sectional area S [m^2].
- Reduced shearing section S_r (SR) [m^2].
- primary moment of inertia (in the local axes of the element) (VIN) [m^4].
- local y-coordinate of the section gravity center (YG) [m].

Parameters for linear elasticity for 3D classical beam elements

Mechanical parameters :

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]

Geometrical parameters :

- Cross-sectional area S [m^2].
- Reduced shearing sections S_2 , S_3 (S2,S3) [m^2].
- torsion inertia VI1 [m^4].
- primary moment of inertia with respect to axes x2 et x3 (VI2,VI3) [m^4].
- Coordinates in the local axes (x2, x3), of the gravity centers of the sections YG, ZG [m].
- Coordinates in the local axes (x2, x3), of the torsion centers of the sections YC, ZC [m].
- VX,VY,VZ : vector defining the first inertia axis (x2).

10.1.2. Linear elasticity with creep (IMOD=2)

For classical beam elements, CESAR proposes a simplified approach to take into account creep (in a linear elastic case) between two successive computations phases. The user inputs the Young's modulus in the actual (final) situation and the initial value (before creep effects take place).

Parameters for linear elasticity with creep for 2D classical beam elements

Mechanical parameters :

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- Initial Young's modulus E (YOUNGREF) [Pa]

Geometrical parameters :

- Cross-sectional area S [m^2].
- Reduced shearing section S_r (SR) [m^2].
- primary moment of inertia (in the local axes of the element) (VIN) [m^4].
- local y-coordinate of the section gravity center (YG) [m].

Parameters for linear elasticity with creep for 3D classical beam elements

Mechanical parameters :

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]
- Initial Young's modulus E (YOUNGREF) [Pa]

Geometrical parameters :

- Cross-sectional area S [m^2].
- Reduced shearing sections S_2 , S_3 (S2,S3) [m^2].
- torsion inertia VI1 [m^4].
- primary moment of inertia with respect to axes x2 et x3 (VI2,VI3) [m^4].
- Coordinates in the local axes (x2, x3), of the gravity centers of the sections YG, ZG [m].
- Coordinates in the local axes (x2, x3), of the torsion centers of the sections YC, ZC [m].
- VX,VY,VZ : vector defining the first inertia axis (x2).

10.2. Multifiber beam elements

IN three-dimension condition, the multifiber beam elements make it possible to take into account:

- several fibers with different mechanical behaviours
- a non linear behaviour for each of the fibers.

The user inpts the number of fibers for the elements of the group, the number of constitutive models necessary to describe the behaviour of the whole set of fibers. Then the user defines the parameters for each of these models, and for each fiber, the relevant geometrical parameters and the constitutive model that must be taken into account (in the list previsouly defined).

The constitutive models for the fibers, associated with the indicator IMODF, are a sub-set of the « classical » models available for elements of families 01/02 (see 3.2) :

- IMODF=1 : linear isotropic elasticity
- IMODF=11 : von Mises without hardening
- IMODF=12 von Mises with hardening
- IMODF=15 parabolic criterion
- IMODF=47 standard Willam Warnke model
- IMODF=48 modified Willam Warnke model

11. Shell elements

The CESAR proposes two types of shell elements: "classic" elements and multilayer elements. These are finite elements of different formulations, but for the user, the distinction is made simply by choosing the model IMOD = 1 for classic shells and IMOD = 2 for multi-layer shells.

11.1. Classic shell elements (IMOD=1)

In CESAR:

- the shell elements with three or four nodes are based on the kinematic hypotheses of the Kirchhoff Love type, which are suitable to model thin shells and plates in pure bending;

- the shell elements with six or 8 nodes are based on Mindlin's assumptions: they give a better account of the transverse shearing, and are adapted to thick structures.

NB: In the current state of code programming, we provide in the ELEM module not only the mechanical characteristics, but also geometrical parameters (for elements of variable thickness from one node to another), and an indicator for the type of results that one wishes (in terms of efforts).

Parameters for the linear elastic model for classic shell elements (IMOD=1)

Mechanical parameters

- density (RO) [kg m^{-3}]
- Young's modulus E (YOUNG) [Pa]
- Poisson's ratio ν (POISS) [-]

Geometrical parameters :

- thickness of the element at the nodes

11.2. Multi-layer shell elements (IMOD=2)

Multilayer shells are elements that represent a stack of layers, which can be associated with a specific constitutive model, defined by the IMODC indicator, to be chosen from the following list:

IMODC = 1 Isotropic linear elasticity.

IMODC = 11 Von mises without hardening.

IMODC = 12 Von Mises with hardening.

IMODC = 15 Parabolic criterion.

IMODC = 47 standard Willam-Warnke model (3 parameters).

IMODC = 48 modified Willam-Warnke model (4 parameters).

These are the same models as for the bulk elements: we will therefore refer to section 3.2. It will be noted that neither the other "classic" models nor the "component" models are available for multilayer shells.

12. Recommendations for the choice of a constitutive model for some simple geotechnical structures

12.1. Shallow foundations

12.2. Deep fondations

12.3. Slope stability

12.4. Tunnel: design of the lining

12.5. Tunnel: settlements induced at the surface

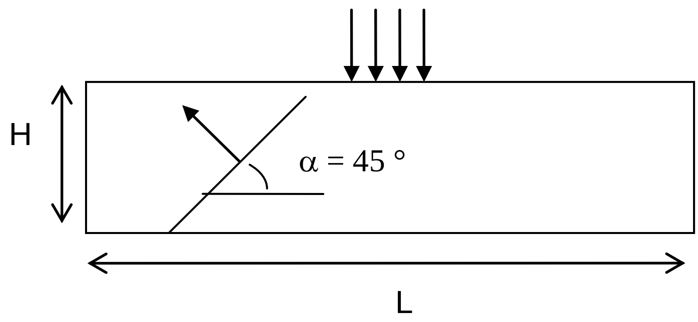
12.6. Retaining wall

13. Examples and elements of validation

This section presents a few examples using some of the constitutive models of CESAR. When possible, some elements are provided regarding the validation of the results.

13.1. Transversely isotropic linear elasticity (IMOD=2)

Consider in plane strain an anisotropic layer of soil, having a dip of 45 degrees with respect to the horizontal, as indicated in FIG. 1. The dimensions of the meshed domain are: L = 16 m; H = 5 m. The mesh consists of 8-node quadrangles, square, 1 m side. The layer is subjected to a uniform pressure p at the surface, over a width of 2 m. The boundary conditions are standard (u = 0 on the two left and right sides of the mesh; v = 0 on the lower face).



The following values are adopted for the mechanical parameters :

E_1 = 40 MPa ; E_2 = 125 MPa ; ν_1 = 0,25 ; ν_2 = 0,25 ; G = 50 MPa ; $\theta = \alpha$ = 45 degrees

The applied pressure is equal to 150 kPa.

The problem can be modelled with the classic model IMOD=2 or with the user defined models.

The data relative to the constitutive model take the following form :

for IMOD=2 :

ground

2 1 .02 40. 125. .25 .25 50. 45.

for IMOD=10000 :

ground

10000 1

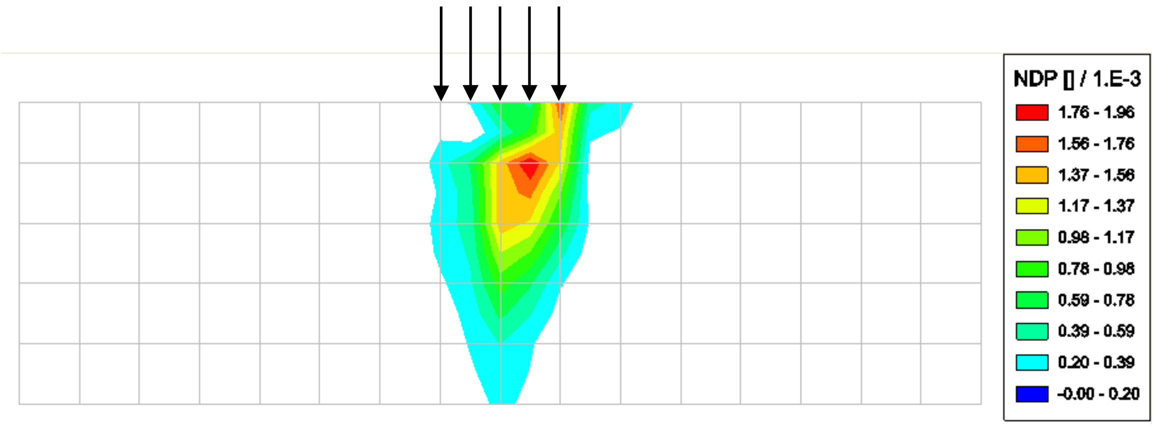
ELAS 2 40. 125. .25 .25 50. 45.

It is recalled that it is possible to dump the content of the rsv4 files in the form of a text file, with the extension .cfor, using the GEFI module with the options "1 1" (to be placed just before MCNL in the datafile).

The agreement between the files cfor obtained is perfect.

Notes: IN the case of the user defined models, the INAT indicator is found on the same line as the value 10000, and that the density value is omitted.

The interest of the user defined models lies in the fact that they allow calculations to be made in three-dimensional conditions, to associate the Mohr-Coulomb model with anisotropic elasticity and work hardening, and to use a anisotropic elasticity with any plastic model.



Contour lines of the norm of the plastic strain : anisotropy induced a non symmetric response, in spite of the fact that the applied load is a uniform pressure.

13.2. Linear isotropic elasticity with moduli depending on z (IELAS=1 or 8)

Consider a horizontal soil layer, of thickness H, resting on a rigid bedrock. We study the deformations which would result from the application of a uniform pressure on the surface $-p\, e_y$ (with p = 100 kPa).

The soil is deformed in oedometric condition, which reduces to a one-dimensional problem. It is assumed that the layer is heterogeneous, and that its modulus depends on the depth. Taking into account the assumptions made, one can use a mesh made up of only one column of superimposed elements: one blocks the horizontal displacement on the vertical sides of the mesh, and the vertical displacement on its base. One takes into account here a mesh comprising twenty quadrangles with eight square nodes of height H / 20. The base of the mesh is in z = 0, and its upper face in z = H.

1/ Modulus varying linearly with depth

We consider the case where the module varies linearly between E1 at the surface (z = H) and E2> E1 at the depth of the substratum (z = 0):

$$E(z) = E_2 + (E_1 - E_2)\, z/H$$

Poisson's ratio is constant and equal to 0.3. The oedometric modulus is defined by:

$$E_{oed}(z) = E(z)\,\frac{(1-\nu)}{(1+\nu)(1-2\nu)} = \alpha + \beta\, z$$

with $\alpha = E_2\frac{(1-\nu)}{(1+\nu)(1-2\nu)}$, $\beta = \frac{E_1 - E_2}{H}\;\frac{(1-\nu)}{(1+\nu)(1-2\nu)}$

The solution is given by:

σ_{zz} = - p ; ε_{zz} = - p / $E_{oed}(z)$; ε_{zz} = - p / $E_{oed}(z)$; ξ_z = - p/ β ln (1 + β z/α)

We compare the results obtained numerically with the IELAS model = 1, and with a mesh in which we associate each element of the mesh with a different group so as to be able to vary (manually) the module from one layer to another (we number the elements from 1 to 20 from bottom to top, and we assign to layer i the module corresponding to the mid-height value: E2 + (E1 - E2) (zi + zi + 1) / 2H).

Take E1 = 100 MPa, E2 = 300 MPa as an example. For the calculation comprising 20 groups of elements, the module in layer i is therefore calculated by E (i) = (295 - 10 * i) (MPa). For the model with variation of the module, all the elements belong to the same group, to which we attribute the following properties:

```
sol
  10000   1
 ELAS 1  300  0.3   -10. 0.
```

The values 300 and 0.3 are the values of E and ν for z = 0, and -10. is the vertical gradient (negative because the modulus decreases as z increases).

The table below gives the computed values and the relative errors with respect to the analytical solution (defined by: relative error = $\frac{\text{numerical solution}}{\text{analytical solution}}$ -1). Both numerical approaches give a very good approximation of the analytical solution, but the relative error on the displacement is smaller (for the same mesh) with the user-defined model.

z (m)	xi (analyt)	xi (ielas=1)	xi (20 groups)	Relative errors	
				ielas =1	20 groups
0	0.0000E+00	0.0000E+00	0.0000E+00		
1	-2.5184E-04	-2.5184E-04	-2.5182E-04	-3.90E-07	-9.57E-05
2	-5.1252E-04	-5.1252E-04	-5.1247E-04	-9.25E-07	-9.85E-05
3	-7.8268E-04	-7.8268E-04	-7.8260E-04	-1.49E-07	-1.04E-04
4	-1.0630E-03	-1.0630E-03	-1.0629E-03	-4.55E-06	-1.08E-04
5	-1.3544E-03	-1.3544E-03	-1.3542E-03	9.54E-07	-1.10E-04
6	-1.6576E-03	-1.6576E-03	-1.6574E-03	1.32E-06	-1.19E-04
7	-1.9738E-03	-1.9738E-03	-1.9736E-03	2.56E-06	-1.24E-04
8	-2.3040E-03	-2.3040E-03	-2.3037E-03	8.51E-07	-1.29E-04
9	-2.6496E-03	-2.6496E-03	-2.6492E-03	1.77E-06	-1.34E-04
10	-3.0120E-03	-3.0120E-03	-3.0116E-03	1.16E-06	-1.42E-04
11	-3.3931E-03	-3.3931E-03	-3.3926E-03	-7.13E-07	-1.51E-04
12	-3.7947E-03	-3.7947E-03	-3.7941E-03	-1.22E-06	-1.62E-04
13	-4.2193E-03	-4.2193E-03	-4.2186E-03	1.51E-09	-1.73E-04
14	-4.6697E-03	-4.6697E-03	-4.6688E-03	-9.27E-07	-1.85E-04
15	-5.1491E-03	-5.1491E-03	-5.1481E-03	-6.49E-07	-2.01E-04

16	-5.6616E-03	-5.6616E-03	-5.6604E-03	-3.21E-07	-2.18E-04
17	-6.2121E-03	-6.2121E-03	-6.2106E-03	2.93E-07	-2.40E-04
18	-6.8067E-03	-6.8067E-03	-6.8049E-03	-1.69E-07	-2.65E-04
19	-7.4531E-03	-7.4531E-03	-7.4509E-03	-1.85E-07	-2.97E-04
20	-8.1611E-03	-8.1611E-03	-8.1584E-03	1.73E-08	-3.37E-04

2 / Modulus varying in z^a

We consider here the same problem, with a slightly different version of the modulus variation:

$$E(z) = E_1 + (E_2 - E_1)\ (1\text{-}z/H)^2$$

E is equal to E_1 at the level of the upper surface (z=H), and to E_2 at the level of the (z=0). Note that, over the interval [0,H], the modulus given by this approach is smaller than the one given by a variation of the modulus linear with depth.

Poisson's ratio is equal to 0,3. The oedometric modulus is given by:

$$E_{oed}(z) = A + B\ (1\text{-}z/H)^2$$

with $A = E_1 \frac{(1-\nu)}{(1+\nu)(1-2\nu)}$, $B = (E_2 - E_1)\frac{(1-\nu)}{(1+\nu)(1-2\nu)}$

The solution is given by:

$$\xi_z = \frac{pH}{\sqrt{AB}}\ [\ \arctan(\sqrt{\frac{B}{A}}\ (1\text{-}z/H)\ )\ -\ \arctan(\sqrt{\frac{B}{A}}\ )]$$

For the mesh with 20 groups, one computes the moduli at the middle of each element, which gives the following values: 290.125, 271.125, 253.125, 236.125, 220.125, 205.125, 191.125, 178.125, 166.125, 155.125, 145.125, 136.125, 128.125, 121.125, 115.125, 110.125, 106.125, 103.125, 101.125 and 100.125 MPa – from the bottom to the top).

For the model with variable moduli, the lines to be written in the datafile are:

```
ground
 10000   1
 ELAS 8  100.  0.5  20.  2.  0.3
```

Note : The modulus is computed according to $E = E_o + k\ (h\text{-}z)^{\alpha}$

The input parameters are in the order : E, k, h, α, ν.

here E_o is equal to the value of E for z=H (soit E_1 = 100 MPa) ; H = 20 m ; α = 2 ; ν = 0,3

and the value of k is computed by : $k = (E_2 - E_1) / H^2$

The table below gives the computed values and the relative errors with respect to the analytical solution (defined by: relative error = $\frac{\text{numerical solution}}{\text{analytical solution}}$ -1). Both numerical approaches give a very good approximation of the analytical solution, but the relative error on the displacement is smaller (for the same mesh) with the user-defined model.

z (m)	xi (analyt)	xi (ielas=8)	xi (20 groups)	Relative errors ielas =8	Relative errors 20 groups
0	0.0000E+00	0.00E+00	0.00E+00		
1	-2.5611E-04	-2.56E-04	-2.56E-04	4.33E-07	-2.34E-04
2	-5.3016E-04	-5.30E-04	-5.30E-04	5.03E-07	-2.33E-04
3	-8.2370E-04	-8.24E-04	-8.24E-04	3.15E-07	-2.34E-04
4	-1.1384E-03	-1.14E-03	-1.14E-03	-5.04E-07	-2.29E-04
5	-1.4759E-03	-1.48E-03	-1.48E-03	2.23E-06	-2.28E-04
6	-1.8382E-03	-1.84E-03	-1.84E-03	-1.33E-06	-2.30E-04
7	-2.2269E-03	-2.23E-03	-2.23E-03	2.09E-06	-2.22E-04
8	-2.6440E-03	-2.64E-03	-2.64E-03	-5.63E-07	-2.16E-04
9	-3.0913E-03	-3.09E-03	-3.09E-03	1.48E-06	-2.06E-04
10	-3.5702E-03	-3.57E-03	-3.57E-03	1.11E-06	-1.92E-04
11	-4.0821E-03	-4.08E-03	-4.08E-03	1.04E-06	-1.78E-04
12	-4.6278E-03	-4.63E-03	-4.63E-03	-6.18E-07	-1.58E-04
13	-5.2076E-03	-5.21E-03	-5.21E-03	-1.36E-07	-1.36E-04
14	-5.8208E-03	-5.82E-03	-5.82E-03	8.95E-07	-1.11E-04
15	-6.4660E-03	-6.47E-03	-6.47E-03	7.28E-07	-8.28E-05
16	-7.1404E-03	-7.14E-03	-7.14E-03	-2.90E-07	-5.21E-05
17	-7.8401E-03	-7.84E-03	-7.84E-03	-5.02E-07	-2.09E-05
18	-8.5602E-03	-8.56E-03	-8.56E-03	-5.40E-07	1.11E-05
19	-9.2945E-03	-9.29E-03	-9.29E-03	1.52E-07	4.10E-05
20	-1.0036E-02	-1.00E-02	-1.00E-02	3.85E-06	6.36E-05

13.3. Use of non linear elastic models

13.3.1. Non linear elasticity of the Cam-Clay model

We only discuss here the role of the elastic part of the model, without involving the criterion of plasticity, and one proposes to model a triaxial test (drained). The initial stress is isotropic and equal to p ° = 200 kPa. The vertical stress is equal to p° + q, and the deviatoric stress q is gradually increased, from 0 to 1 MPa, in increments of 50 kPa.

We recall that the nonlinear elasticity model associated with the Cam-Clay model is isotropic, with G = cte and $K = p(1+e_o)/\kappa = \alpha\, p$.

It is easy to see that the volume deformation is given by:

$$d\varepsilon_v = -\,dp/K = \frac{dp}{K} = \frac{dp}{\alpha p} \Rightarrow \varepsilon_v = -\frac{\ln(1+q/3p°)}{\alpha}$$

Moreover, one can derive the axial strain as follows:

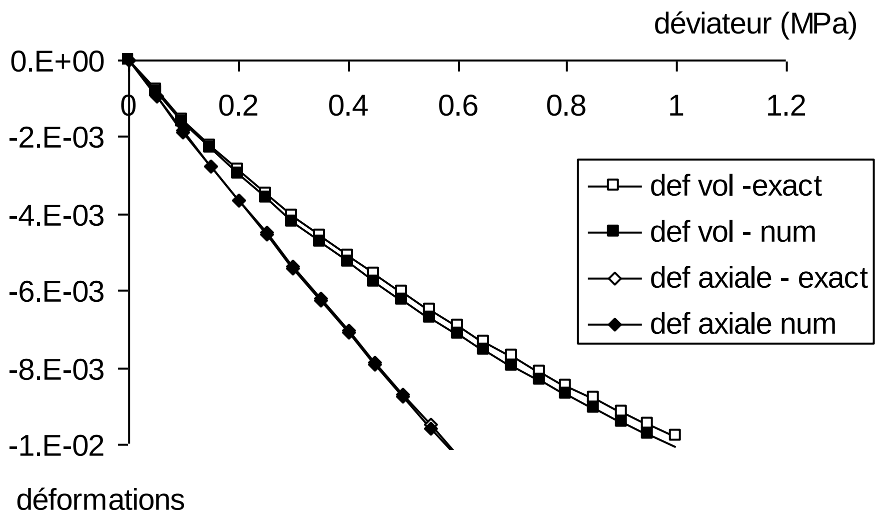
$E = 9KG/(3K+G)$

$$d\varepsilon_1 = -dq/E = -3\frac{dp}{E} = -\frac{3K+G}{3KG}\,dp = -\frac{3\alpha p+G}{3\alpha pG}\,dp \;\Rightarrow \varepsilon_1 = -\frac{q}{3G} - \frac{\ln(1+q/3p^{\circ})}{3\alpha}$$

The analytical and numerical results are shown in the figure below for G=25 MPa, e_o = 1., κ = 0.02 (with K_{min}=0.1 MPa).

The lines corresponding to the constitutive model in the data file are:

```
sol
10000  2
ELAS   3   25.    1.   0.02  0.01
```



Notes :

1 / The non linearity of the elastic part of the model is treated by to updating the moduli at the beginning of each increment: we therefore do not exactly verify the law of behavior (each increment is carried out in a single iteration). We could imagine a more complete programming - at least for the elastic models whose formulation makes it possible to calculate the error committed on the constitutive model.

13.3.2. Non linear elasticity of the “Fahey and Carter” model

The introduction of this type of elastic model was undertaken during the PhD of S. Coquillay (2005). If we fit the parameters of the Mohr-Coulomb model, with a linear elasticity, on the initial slope of triaxial tests, we obtain a too steep response at the scale of the geotechnical structure: modeling loading tests on experimental shallow foundations at Labenne had highlighted this problem. Conversely, if we look for values that allow us to find the results of tests on structures, the modules obtained are too small to correctly represent laboratory tests.

One presents here an example of calibration of the model on a triaxial test carried out on sand of Labenne: the model proposed by Fahey and Carter makes it possible to represent a little better the curve q- ε_1 during shearing, and in particular to better reproduce its curvature (CIS4 essay from Sophie Coquillay's thesis, taken from an internal LCPC report (Gestin, 1989)).

Test with initial isotropic stress 300 kPa; strain-driven simulation.

Parameters of the criterion and the flow law: c = 1 kPa; φ = 35.5° ; ψ = 8°

Parameters of the Mohr-Coulomb model with linear elasticity:

We performed two calculations, the first with E = 96 MPa (which allows to find the initial slope of the curve q-ε_1) and the second with E = 32 MPa (which allows to find an acceptable strain when q reaches 90% of its ultimate value). In both cases, the Poisson's ratio is 0.3, and the corresponding data is therefore:

case 1 :

sand

10 2 0. 96.e3 0.3 1. 35.5 8.5

Case 2 :

sand

10 2 0. 32.e3 0.3 1. 35.5 8.5

Parameters for the Fahey-Carter based elastic model (IELAS=4)

257/5000

We associate the elastic model IELAS=4, with a Mohr Coulomb criterion (ICRIT = 4) and a Mohr-Coulomb flow potential (but the dilatancy angle is different from the friction angle). The parameters for the Fahey and Carter model are as follows:

n = 0,5 ; ν°=0,25 ; f=0,8 ; g = 1,8 ; C = 225 ; p_{ref} = 100 kPa ; c = 1 kPa ; φ = 35.5°

remember that in this model, elastic moduli depend on the strength parameters. The input in the datafile is as follows:

sand

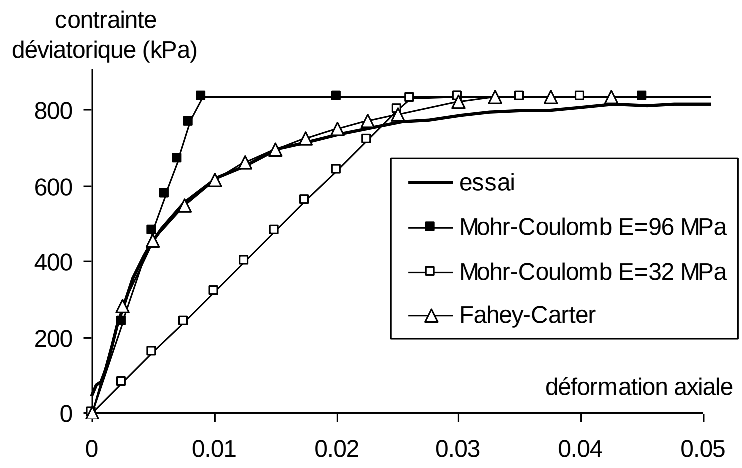
10000 2

ELAS 4 0.5 0.25 0.8 1.8 225 100. 1. 35.5

CRT 4 1. 35.5

POT 4 0. 8.5

The results are compares on the experiments on the following figure.



The following figure shows the results obtained with the simplified version of the Fahey-Carter model (IELAS=7).

Parameters for the modified Fahey-Carter based elastic model (IELAS=7)

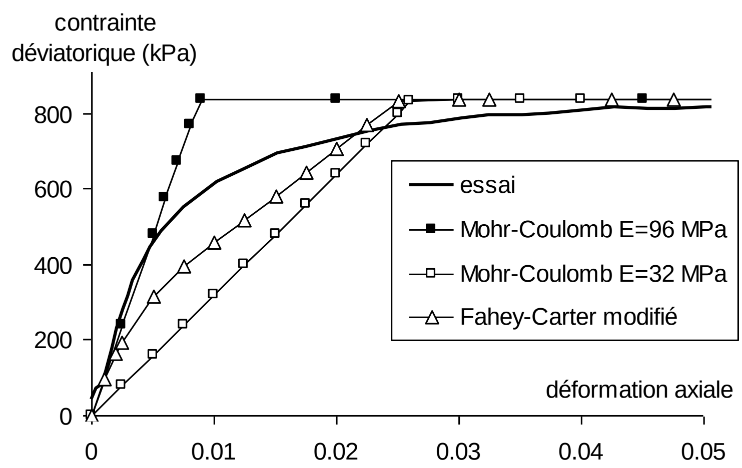
sand

10000 2

ELAS 7 38.4e3 300. 0.5 10.e3 250. 0.25

CRT 4 1. 35.5

POT 4 0 8.5



The non linear elastic models fit much better the experimental curves than the linear elastic models.

Note however that the model obtained with the elastic law of Fahey and Carter (IELAS = 4) and the criterion of Mohr Coulomb presents several weak points. First, it involves a large number of parameters whose identification from tests is difficult (see Coquillay, 2005). On the other hand, the model is perfectly plastic: the module is therefore the same in loading and unloading, and the axial deformation for which one passes from the elastic contracting phase to the dilating phase corresponds to the maximum value of the deflector.

The modified version depends on a smaller number of parameters, and the modules do not depend on the resistance characteristics. This model allows to reproduce the initial stiffness and to calibrate the axial deformation for a given value of the deflector, but not to reproduce the shape of the curve as well as with the original model.

13.4. Computation in undrained condition

The user defined models make it possible to perform calculations in undrained condition. The principle is to penalize, for the assembly of the rigidity method, the volumetric strain by taking into account the compressibility of the interstitial fluid. The treatment of plasticity rests well on a calculation in effective stresses.

There is a particularity here: the loadings applied are given in total stresses. The results of computation are in effective stresses (which explains that one does not find the stresses which one applies on certain parts of the contour).

The analysis of the results must be done with caution, because the treatment is done group by group (we can have low permeable groups undrained and permeable groups which remain in drained condition): we can therefore have vector discontinuities- constraint for the computed stress field.

We present the result of the simulation of an undrained triaxial test, with the modified Cam-Clay model. The material properties are:

Linear isotropic elasticity : E = 50 MPa ; ν = 0,3

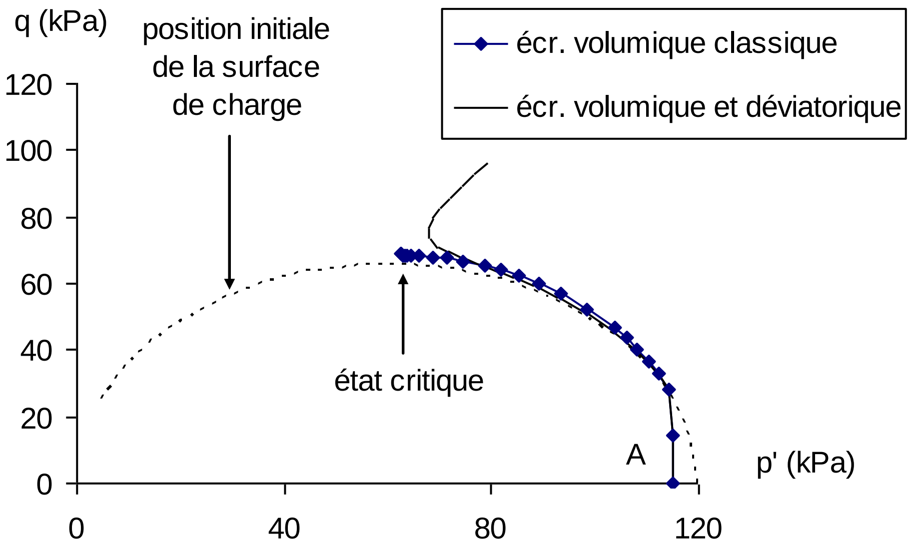
Parameters for the criterion: M = 1,1 ; $p_c°$= 120 kPa

Parameters for the hardening law: λ = 0,02 ; κ = 0,08 ; e_o = 0,9

Starting from an isotropic stress state p ’= 115 kPa, a vertical displacement is applied downwards on the upper face of the mesh. The calculation is carried out in undrained condition, adopting for water a compression modulus equal to 2000 MPa and for the porosity the value 0.3.

fichier : triax_ccm6.data

The following figure represents the stress path in the plane (p ’, q). We note that the state of stresses, starting from point A, rises first along a vertical: during the elastic regime, the variation in volume is negligible and it is the same for that of p’. The stress state reaches the yeild surface and the point then goes up following the load surface, to the critical state line: with this model, the material has a clearly identified shear limit in undrained condition. It will be noted that the size of the elastic domain increases a little during the shearing, so that one does not remain exactly on the initial surface.



For illustration purpose, the stress path obtained also has been shown if a deviatoric component is taken into account in the hardening (IECR = 3), by modifying the material data.

With this type of work hardening, and according to the valuesof the parameters, one can obtain a path of stresses different on the qualitative level, and an inversion of the direction of variation of p' during shearing. On the other hand, the deviatoric stress continues to increase: there is no longer necessarily a limit to the undrained shear strength.

Notes:

1) The Egg Cam-Clay model is a variant in which one can modify the form of the criterion, on the one hand, to authorize tensile stresses, on the other hand, and finally, to adopt a volume hardening law only in the contracting party of the criterion.

2) The use of computations in undrained condition for the evaluation of the short-term behavior of works requires particular precautions: if the choice of the criterion of plasticity is not appropriate, the model can lead to an infinite resistance to shearing in undrained condition. This is particularly the case if we use the Mohr-Coulomb model. In plastic regime, the stress path goes up indefinitely along the rupture line. As a result, this type of model is not suitable for the assessment of the stability of a structure in the short term.

13.5. Anisotropic plasticity criterion

To be completed

13.6. Cyclic simulations with non linear kinematic hardening models (ICRIT=25,27)

To account for a progressive accumulation of plastic strain under repeated loadings, a non linear kinemartic hardening has been implemented. In this approach, the yield surface is translated in the stress space:

$$f(\underline{\underline{\sigma}}) = F(\underline{\underline{\sigma}}-\underline{\underline{X}})$$

where $\underline{\underline{X}}$ is the hardening parameter and F a classic yield function : that of the von Mises model for ICRIT=25 and that of the Drucker-Prager model for ICRIT=27. The accumulation of strain is obtained using a hardening law proposed by Chaboche or by Armstrong and Fredericks:

$$\dot{\underline{\underline{X}}} = 2/3\ C\ \dot{\underline{\underline{\varepsilon}}}^p - D\ \underline{\underline{X}}\ \dot{\xi}$$

where C is a reference pressure and D a dimensionless quantity, whereas $\dot{\xi} = (2/3\ \dot{\underline{\underline{\varepsilon}}}^p : \dot{\underline{\underline{\varepsilon}}}^p)^{1/2}$. For the flow rule, the plastic potential is computed for the translated variable:

$$g(\underline{\underline{\sigma}}) = G(\sigma-X)$$

with

$$G(\sigma) = \sqrt{\frac{1}{2}\underline{\underline{s}}:\underline{\underline{s}}} + \beta\ \mathrm{tr}\ \underline{\underline{\sigma}}$$

It is possible to integrate the model equations analytically for a triaxial test of monotonic compression. Starting from a supposedly isotropic stress state $\underline{\underline{\sigma}}° = -\sigma_3\underline{\underline{1}}$, we keep the confining stress σ_3 constant, and we gradually increase the (absolute value of) vertical stress. We set $\sigma 1 < \sigma 3 < 0$ and

$$\underline{\underline{\sigma}} = \sigma_1\underline{e}_1\otimes\underline{e}_1 + \sigma_3(\underline{e}_2\otimes\underline{e}_2 + \underline{e}_3\otimes\underline{e}_3) \quad ; \quad q = \sigma_3 - \sigma_1$$

$$\underline{\underline{X}} = X_1\underline{e}_1\otimes\underline{e}_1 + X_3(\underline{e}_2\otimes\underline{e}_2 + \underline{e}_3\otimes\underline{e}_3) \quad ; \quad X_3 = X_1 + x.$$

It can be shown that the deviatoric stress and the axial plastic strain are related to each other by :

$$q = q^{el} + (q^{max} - q^{el})\ [\ 1 - \exp(\gamma\varepsilon_1^p)\ ]$$

where q^{el} is the elastic limit (the largest value of q for which the material remains in the elastic domain) and q^{max} the ultimate value of the deviatoric stress. One gets :

$$q^{el} = \frac{k - 3\ \alpha\sigma_3}{1/\sqrt{3} - \alpha} \quad ; \quad q^{max} - q^{el} = \frac{C}{D}\frac{1+6\alpha\beta}{\sqrt{3\ (1+6\ \beta^2)}}\ \frac{1}{1/\sqrt{3} - \alpha} \quad ; \quad \gamma = \frac{D\sqrt{2\ \beta^2 + 1/3}}{1/\sqrt{3} - \beta}$$

Also, the following relation can be established between the volumetric plastic strain and the axial plastic strain :

$$\varepsilon_v^p = \varepsilon_1^p + 2\ \varepsilon_3^p = 3\ \dot{\lambda}\ \beta = \frac{3\beta}{\beta - 1/\sqrt{3}}\ \varepsilon_1^p$$

These relations can be used to derive the equation of the q-ε_1 and ε_v-ε_1 curves:

$$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p = -\frac{q}{E} - \frac{1}{\gamma}\ln\left(\frac{q^{max} - q^{el}}{q^{max} - q}\right)$$

$$\varepsilon_v = \varepsilon_v^e + \varepsilon_v^p = -\frac{1-2\nu}{E}\ q + \frac{3\beta}{\gamma(1/\sqrt{3}-\beta)}\ln\left(\frac{q^{max} - q^{el}}{q^{max} - q}\right)$$

It is observed in particular that the change of sign of the variation in volume occurs for a value of the deviator q^{cr} strictly less than q^{max} (which constitutes an improvement compared to the perfectly plastic nonlinear elastic model).

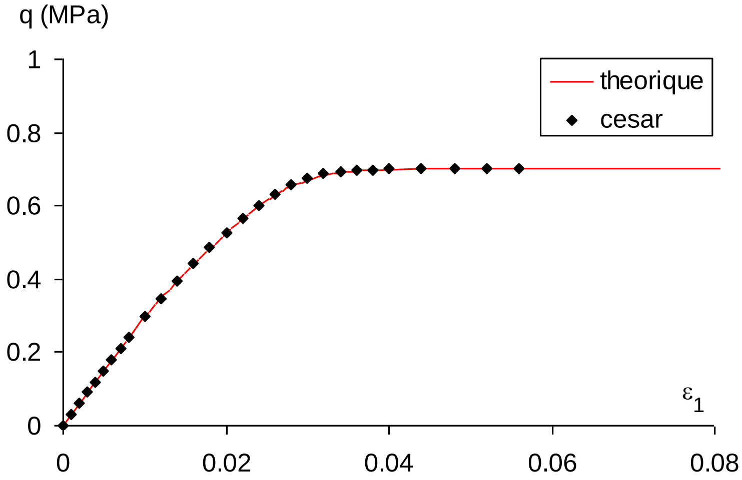
$$q^{cr} = q^{max} - \frac{3\beta\ E}{\gamma(1/\sqrt{3}-\beta)\ (1-2\nu)} = q^{max} - \frac{3\beta\ E}{(1-2\nu)\ D\sqrt{2\ \beta^2 + 1/3}}$$

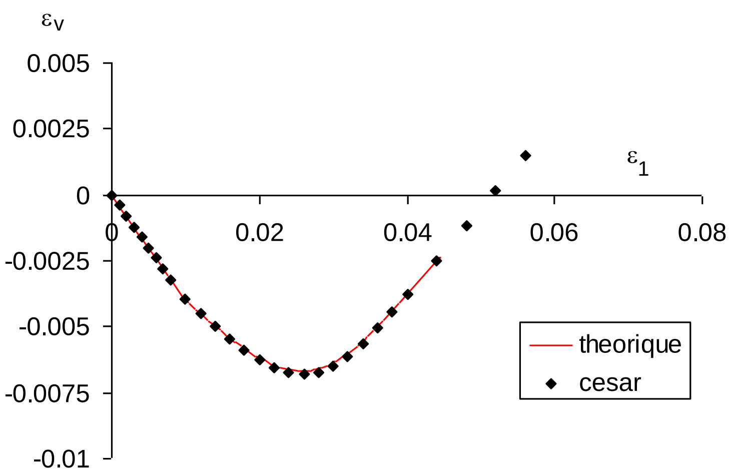
These relations make it possible to validate the implementation, at least in the monotonous case. Mestat (2010) also gave the equation of the curve q-ε_1 in the case of an unloading from a charge level lower than q^{max}.

The figures below show the results obtained numerically for E= 30 MPa, ν=0.3, c=100 kPa ; φ = 25 ° ; ψ = 10° ; C=90 MPa ; D=300.

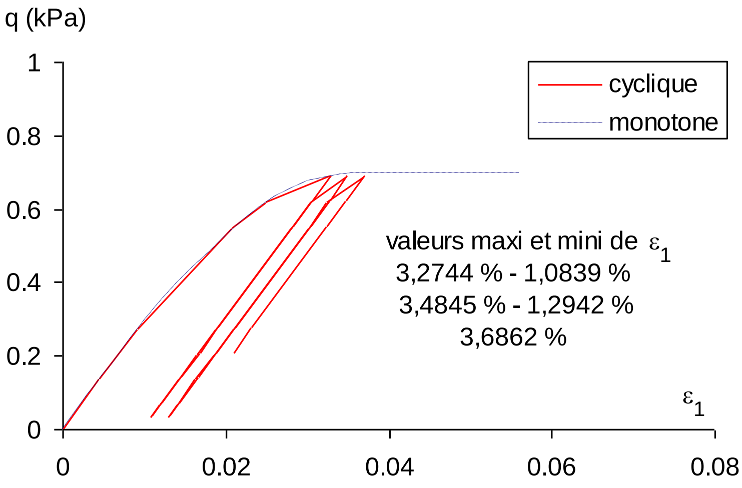
The computation simulates a triaxial test for an initially isotropic stress equal to 100 kPa, then a displacement is prescribed (3% of vertical strain in 30 increments, then 9 additional increments of 0,1%).

The ultimate value of the diverter is 700.86 kPa according to the above formula: the calculation gives the same value. The figure below shows the comparison between computation and analytical expressions, for the two curves q-ε_1 and ε_v-ε_1.





We can also represent a cyclic test, in which the deviatoric stress, initially zero, is gradually increased to 690 kPa; one carries out cycles between this value and a deflector of 34.5 kPa, for the same values of the parameters.



The results show a progressive accumulation of plastic strain as the number of cycles increases.

13.7. Simulations with two plastic mechanims

13.7.1. Cap in compression

We consider a triaxial test for a model combining two independent elastoplastic mechanisms. Such a combination is possible with the user defined models, provided that only smooth yield surface are chosen (which excludes the Mohr-Coulomb criterion and that of Hoek and Brown).

We can for example consider the combination of a parabolic criterion (without hardening) and a Cam-Clay type criterion, with the usual volume hardening. The elastic parameters and the two plastic mechanisms are as follows:

$E = 100$ MPa ; $\nu = 0{,}3$

$R_c = 23$ kPa ; $R_t = 3$ kPa

$M = 1$; $p_c° = 150$ kPa ; $e_o = 1.08$, $\lambda = 0{,}15$ et $\kappa = 0{,}02$.

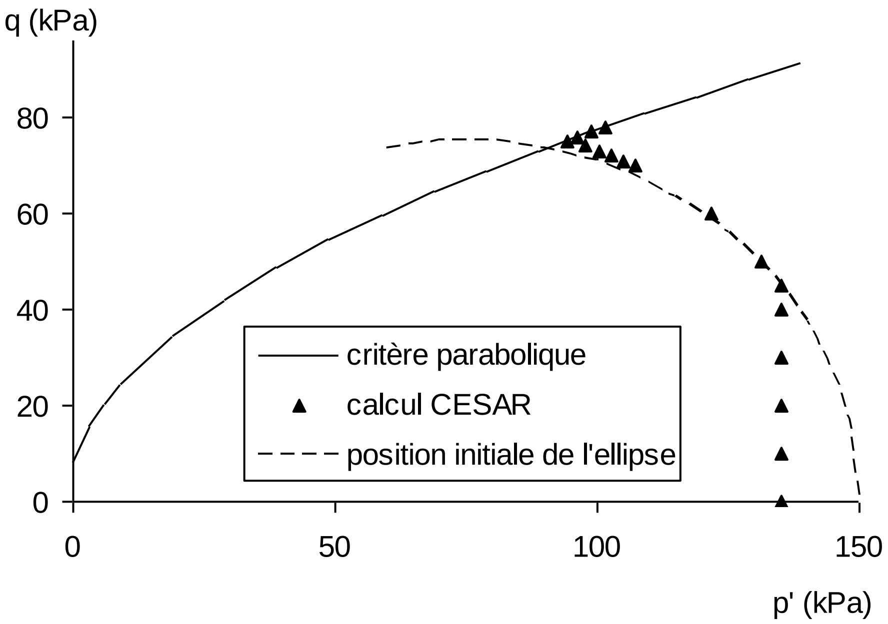
We simulate a triaxial test, for an initial isotropic stress of 135 kPa. The data are as follows:

```
clay
10000   2
ELAS 0   30.e3 0.3
CRT  7   22.75    3.
CRT2  6   1.   150.
ECR2  1   .15  0.02  1.08
NDR
1000e3  .3
```

The stress path obtained in the plane (p ', q) is represented in the following figure, by the succession of triangular symbols. Starting from the initial isotropic stress state, we start with a path at constant p '; when the stress state reaches the load surface of the Cam-Clay model (whose initial position is shown in dotted lines), we go up on this surface, the mean stress decreases and the deviatoric stress increases.

When the stress path reaches the second surface that corresponds to the parabolic criterion, shown in solid lines), the direction of variation of the mean stress is reversed.

Note : The computation is stress-driven here. Convergence when reaching the second surface (the parabolic criterion) is very slow (here we adopted a tolerance of 0.01 and fixed the maximum number of iterations at 2000); it is simpler in this case to perform a displacement-driven simulation.



13.7.2. Fractured rock mass

Another possible use of the combination of two plastic mechanisms has been proposed by A. Pouya to represent the behaviour of a rock mass with a family of fractures. The approach consists in combining an isotropic criterion for the strength properties of the intact rock mass (for instance the Drucker Prager model), with an anisotropic model (such as the directional criterion) to take into account the strains that can occur inside the fractures.

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15. APPENDICES

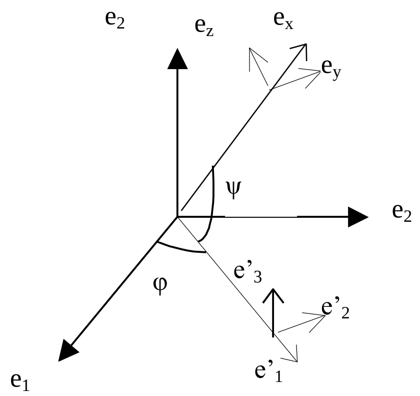
15.1. Anisotropic plasticity criteria: orientation of the material axes

An anisotropic material has different properties depending on the directions of space. It is necessary to mark the orientation of the material in relation to that of the working axes. We denote (e1, e2, e3) the frame in which the mesh is defined and (ex, ey, ez) the frame defining the directions of orthotropy of the material.

Special case of "transverse isotropic" or "orthotropic revolution" models.

In this case, the properties of the material have a symmetry of revolution around a particular axis, the direction of which depends on two angles in 3D.

We define the orientation of the local coordinate system using two angles as follows: this change of coordinate system is familiar to the users for the definition of boundary conditions.



The local axes (e_x, e_y, e_z) is the image of the axes (e_1, e_2, e_3) by the composition of the two angles φ and ψ. We have :

$$\underline{e}'_1 = \cos\varphi\,\underline{e}_1 + \sin\varphi\,\underline{e}_2 \quad ; \quad \underline{e}'_2 = -\sin\varphi\,\underline{e}_1 + \cos\varphi\,\underline{e}_2 \quad ; \quad \underline{e}'_3 = \underline{e}_3$$

and

$$\underline{e}_x = \cos\psi\,\underline{e}'_1 + \sin\psi\,\underline{e}'_3; \quad \underline{e}_y = \underline{e}'_2; \quad \underline{e}_z = -\sin\psi\,\underline{e}'_1 + \cos\psi\,\underline{e}'_3$$

It follows that

$$\begin{bmatrix}\underline{e}_x\\ \underline{e}_y\\ \underline{e}_z\end{bmatrix} = R.\begin{bmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{bmatrix} \quad \text{with} \quad R = \begin{bmatrix}\cos\psi\cos\varphi & \sin\varphi\cos\psi & \sin\psi\\ -\sin\varphi & \cos\varphi & 0\\ -\cos\varphi\sin\psi & -\sin\varphi\sin\psi & \cos\psi\end{bmatrix}$$

$$\begin{bmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{bmatrix} = T\begin{bmatrix}\underline{e}_x\\ \underline{e}_y\\ \underline{e}_z\end{bmatrix}, \quad \text{with } T = R^{-1} = \begin{bmatrix}\cos\psi\cos\varphi & -\sin\varphi & -\cos\varphi\sin\psi\\ \sin\varphi\cos\psi & \cos\varphi & -\sin\varphi\sin\psi\\ \sin\psi & 0 & \cos\psi\end{bmatrix}$$

NB : $R^{-1} = {}^tR$ since R is the matrix of a rotation.

Orientation of the material in the general case (ICRIT=22, 23, 28, 29)

In the most general case, there is no "isotropy plane": it is not enough to locate the direction of a particular material axis, but it is necessary to define the orientation of a reference set of unit vectores with respect to another, which requires three angles instead of two. One of the classic methods for describing a change of reference in mechanics consists in using the Euler angles. The transition from the initial reference to the final reference is made by composing three rotations:

- a rotation of axis e3 and angle Ψ changes from (e1, e2, e3) to (u, v, e3)

- a rotation of axis u and angle Θ changes from (u, v, e3) to (u, w, ez)

- a rotation of axis ez and angle Φ changes from (u, w, ez) to (ex, ey, ez)

The angle Ψ is called the precession angle, Θ is the nutation angle, and Θ the intrinsic rotation angle.

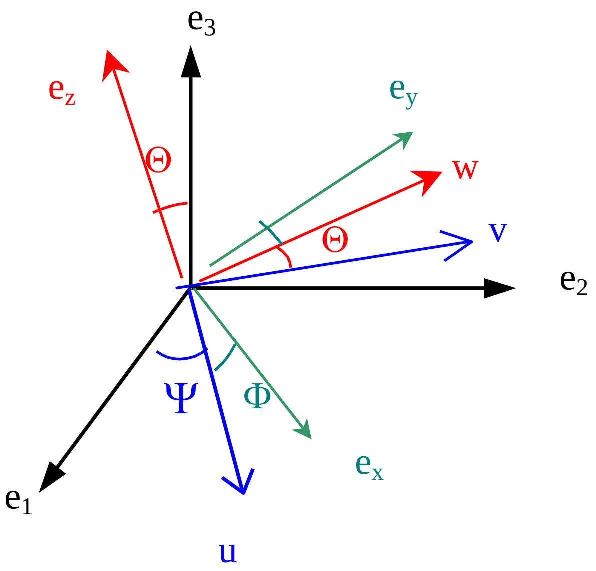


Figure 5 – Euler's angles

In anisotropic models, the criterion depends on the components of the stress tensor in the local coordinate system (ex, ey, ez). It is therefore necessary to write the components of the vectors of the global coordinate system (e1, e2, e3) in the base of the vectors of the local coordinate system (ex, ey, ez). Let

$$u = \cos\Psi\, e_1 + \sin\Psi\, e_2$$

$$v = -\sin\Psi\, e_1 + \cos\Psi\, e_2$$

then

$$w = \cos\Theta\; v + \sin\Theta\; e_3 = -\cos\Theta\sin\Psi\, e_1 + \cos\Theta\cos\Psi\, e_2 + \sin\Theta\; e_3$$

$$e_z = -\sin\Theta\; v + \cos\Theta\; e_3 = \sin\Theta\sin\Psi\, e_1 - \sin\Theta\cos\Psi\, e_2 + \cos\Theta\; e_3$$

and eventually

$$e_x = \cos\Phi\, u + \sin\Phi\, w$$

$$e_y = -\sin\Phi\, u + \cos\Phi\, w$$

We have : $\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = R.\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$

with

$$R = \begin{bmatrix} \cos\Phi\cos\Psi - \sin\Phi\cos\Theta\sin\Psi & \cos\Phi\sin\Psi + \sin\Phi\cos\Theta\cos\Psi & \sin\Theta\sin\Phi \\ -\sin\Phi\cos\Psi - \cos\Phi\cos\Theta\sin\Psi & -\sin\Phi\sin\Psi + \cos\Phi\cos\Theta\cos\Psi & \sin\Theta\cos\Phi \\ \sin\Theta\sin\Psi & -\sin\Theta\cos\Psi & \cos\Theta \end{bmatrix}$$

It follows that:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = T \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}, \text{ with } T = R^{-1} = {}^tR \text{ since R is a rotation matrix.}$$

The previous case is a particular case of the use of Euler angles, with:

$$\Psi = \varphi - \pi/2 \; ; \quad \Theta = \psi \quad ; \quad \Phi = \pi/2$$

Expression of the stress tensor in (e_x, e_y, e_z)

$$\underline{\underline{\sigma}} = \sigma_{11}\underline{e}_1 \otimes \underline{e}_1 + \sigma_{22}\underline{e}_2 \otimes \underline{e}_2 + \sigma_{33}\,\underline{e}_3 \otimes \underline{e}_3$$
$$+ \sigma_{12}\,(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) + \sigma_{23}\,(\underline{e}_2 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_2) + \sigma_{13}\,(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1)$$

The expressions of $e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_3 \otimes e_3$, $e_1 \otimes e_2$, $e_2 \otimes e_3$, $e_1 \otimes e_3$ in (e_x, e_y, e_z) are given by:

$$e_1 \otimes e_1 = T_{1k}\, e_k \otimes \; T_{1l}\, e_l \;=\; T_{1k} T_{1l} \;\; e_k \otimes e_l$$

The matrix of $e_1 \otimes e_1$ in (e_x, e_y, e_z) has for general term $(T_{1k}\, T_{1l})_{k,l=x,y,z}$

And similarly:

$$e_2 \otimes e_2 = (T_{2k}\, T_{2l}\, e_k \otimes e_l)_{\,k,l=x,y,z} \qquad e_3 \otimes e_3 = (T_{3k}\, T_{3l}\, e_k \otimes e_l)_{\,k,l=x,y,z}$$

$$e_1 \otimes e_2 + e_2 \otimes e_1 = (T_{1k}\, T_{2l}\;\; e_k \otimes e_l + T_{1l}\, T_{2k}\, e_k \otimes e_l)_{\,k,l=x,y,z}$$

$$e_2 \otimes e_3 + e_3 \otimes e_2 = (T_{2k}\, T_{3l}\;\; e_k \otimes e_l + T_{2l}\, T_{3k}\, e_k \otimes e_l)_{\,k,l=x,y,z}$$

$$e_1 \otimes e_3 + e_3 \otimes e_1 = (T_{1k}\, T_{3l}\;\; e_k \otimes e_l + T_{1l}\, T_{3k}\, e_k \otimes e_l)_{\,k,l=x,y,z}$$

$$\sigma = \sigma_{11}\,(T_{1k}\, T_{1l}\, e_k \otimes e_l)_{k,l} + \sigma_{22}\,(T_{2k}\, T_{2l}\, e_k \otimes e_l)_{\,k,l} + \sigma_{33}\,(T_{3k}\, T_{3l}\, e_k \otimes e_l)_{\,k,l} \; +$$
$$\sigma_{12}\,(T_{1k}\, T_{2l}\, e_k \otimes e_l + T_{1l}\, T_{2k}\, e_k \otimes e_l)_{\,k,l} \; + \sigma_{23}\,(T_{2k}\, T_{3l}\;\; e_k \otimes e_l + T_{2l}\, T_{3k}\, e_k \otimes e_l)_{\,k,l} \; +$$
$$\sigma_{13}\,(T_{1k}\, T_{3l}\, e_k \otimes e_l + T_{1l}\, T_{3k}\, e_k \otimes e_l)_{\,k,l}$$

Denoting by [σ] the vector of the components of the stress tensor in the local axes, [Σ] the vector of the components in the global axes, and Q the matrix above:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} \quad ; \qquad [\Sigma] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \;;$$

The following holds:

$$[\sigma] = Q\,[\Sigma]$$

with

$$Q = \begin{bmatrix} T_{11}\,T_{11} & T_{21}\,T_{21} & T_{31}\,T_{31} & 2\,T_{11}\,T_{21} & 2\,T_{21}\,T_{31} & 2\,T_{11}\,T_{31} \\ T_{12}\,T_{12} & T_{22}\,T_{22} & T_{32}\,T_{32} & 2\,T_{12}\,T_{22} & 2\,T_{22}\,T_{32} & 2\,T_{12}\,T_{32} \\ T_{13}\,T_{13} & T_{23}\,T_{23} & T_{33}\,T_{33} & 2\,T_{13}\,T_{23} & 2\,T_{23}\,T_{33} & 2\,T_{13}\,T_{33} \\ T_{12}\,T_{11} & T_{21}\,T_{22} & T_{31}\,T_{32} & T_{11}\,T_{22} + T_{12}\,T_{21} & T_{21}\,T_{32} + T_{22}\,T_{31} & T_{11}\,T_{32} + T_{12}\,T_{31} \\ T_{31}\,T_{12} & T_{23}\,T_{22} & T_{32}\,T_{33} & T_{12}\,T_{23} + T_{13}\,T_{22} & T_{22}\,T_{33} + T_{23}\,T_{32} & T_{12}\,T_{33} + T_{13}\,T_{32} \\ T_{13}\,T_{11} & T_{23}\,T_{21} & T_{31}\,T_{33} & T_{11}\,T_{23} + T_{13}\,T_{21} & T_{21}\,T_{33} + T_{23}\,T_{31} & T_{11}\,T_{33} + T_{13}\,T_{31} \end{bmatrix}$$

Computation of the partial derivatives of the yield function

The criterion is given by :

$$f(\sigma) = \frac{3}{2}\,{}^{T}[\sigma]\;P\;[\sigma]\; - \bar{\sigma}^2 = \frac{3}{2}\,{}^{T}[\Sigma]\;{}^{T}Q\;P\;Q\;[\Sigma]\; - \bar{\sigma}^2$$

It follows that:

$$df = \frac{\partial f}{\partial \Sigma}\,d\Sigma = \frac{3}{2}\{{}^{t}[d\Sigma].{}^{t}QPQ.[\Sigma] + {}^{t}[\Sigma].{}^{t}QPQ.[d\Sigma]\} = \frac{3}{2}\{\,{}^{t}[\Sigma].{}^{t}Q{}^{t}PQ.\,[d\Sigma] + {}^{t}[\Sigma].{}^{t}QPQ.\,[d\Sigma]\,\}$$

Since P is symmetric:

$$df = \frac{3}{2}\{\,2\,{}^{t}[\Sigma].\,{}^{t}QPQ.\,[d\Sigma]\,\}$$

which yields $\frac{\partial f}{\partial \Sigma} = 3\,{}^{t}[\Sigma].\,{}^{t}QPQ = 3\,{}^{t}QPQ\,[\Sigma]$

or else, in terms of components in the local axes:

$$\frac{\partial f}{\partial \Sigma} = 3\,{}^{t}[\sigma]\,PQ = 3\,{}^{t}Q\,P[\sigma]$$

15.2. Anisotropic elastoplastic model with two types of hardening: S-CLAY1

This model was developed for natural clays, by Wheeler et al (2003). It is an elastoplastic model defined by an anisotropic criterion. The construction starts from experimental observations in the context of the triaxial test, then proposes an extension to the most general three-dimensional case.

In the context of the triaxial test, the S-CLAY 1 model defines the yield surface by:

$$f = (q - \alpha p)^2 - (M^2 - \alpha^2)\,(p_m - p)\,p = 0$$

where p and q are defined by : $q = \sigma_v - \sigma_h$ et $p = (\sigma_v + 2\,\sigma_h)/3$. These quantities are not the classic invariant: they are defined with reference to specific directions in space.

Figure 6 – Shape of the yield surface of the S-CLAY1 model for $\alpha = 0{,}4$; $p_m = 2{,}5$; $M=1$

It is important to emphasize that one cannot correctly represent the anisotropy of plastic properties if one formulates the criterion by identifying p and q with classical invariants, since one loses the information on the orientation of the directions principal of the stress tensor. In the general three-dimensional framework, Wheeler et al (2003) propose the following formulation:

$$f = 3/2\ (\underline{\underline{s}}-p\ \underline{\underline{a}}^*) : (\underline{\underline{s}}-p\ \underline{\underline{a}}^*) - (M^2 - 3/2\ \underline{\underline{a}}^* : \underline{\underline{a}}^*)\ (p_m - p)\ p$$

where $\underline{\underline{s}}$ is the deviatoric part of $\underline{\underline{\sigma}}$ and $\underline{\underline{a}}^*$ a deviaotric tensor playing the role of hardening parameter. The model has two hardening parameters: $\underline{\underline{a}}^*$ is a tensor while p_m is a scalar.

Le paramètre p_m contrôle la taille de la surface de charge et $\underline{\underline{a}}^*$ son orientation dans l'espace des contraintes.

For parameter p_m, Wheeler et al (2003) propose a classical hardening law :

$$p_m = p_m{}^\circ \exp\left(- \frac{(1+e_o)\varepsilon_v{}^p}{\lambda - \kappa} \right)$$

where e_o is the void ratio, λ and κ the slopdes of the e – ln p curve for an isotropic compression test.

Derivatives of the criterion

$$f = 3/2\ (\underline{\underline{s}}-p\ \underline{\underline{a}}^*) : (\underline{\underline{s}}-p\ \underline{\underline{a}}^*) - (M^2 - 3/2\ \underline{\underline{a}}^* : \underline{\underline{a}}^*)\ (p_m - p)\ p$$

which can also be written as

$$f = 3/2\ \underline{\underline{s}}:\underline{\underline{s}} - 3\ p\ \underline{\underline{a}}^*:\underline{\underline{\sigma}} - M^2\ p\ (p_m - p) + 3/2\ \underline{\underline{a}}^*:\underline{\underline{a}}^*\ p\ p_m$$

Then we get

$$\frac{\partial f}{\partial \underline{\underline{\sigma}}} = 3\ \underline{\underline{s}} - 3\ p\ \underline{\underline{a}}^* + (\underline{\underline{a}}^* : \underline{\underline{s}})\ \underline{\underline{1}} + M^2\ (p_m - 2p)/3\ \underline{\underline{1}} - 1/2\ \underline{\underline{a}}^*:\underline{\underline{a}}^*\ p_m\ \underline{\underline{1}}$$

Computing the hardening modulus H requires to compute the derivatives of f with respect to $\underline{\underline{a}}^*$ and p_m.

$$H\ d\lambda = - \frac{\partial f}{\partial \underline{\underline{a}}^*} : d\underline{\underline{a}}^* - \frac{\partial f}{\partial p_m} : dp_m$$

with $$\frac{\partial f}{\partial \underline{\underline{a}}^*} = 3\ p\ (p_m \underline{\underline{a}}^* - \underline{\underline{s}})$$

$$\frac{\partial f}{\partial p_m} = -\left(M^2 - 3/2\ \underline{\underline{a}}^*:\underline{\underline{a}}^* \right) p$$

The authors propose to take :

$$d\underline{\underline{a}}^* = \mu\ [\ \{ \chi_v - \underline{\underline{a}}^* \} < d\varepsilon_v^p > + \beta\ \{ \chi_d - \underline{\underline{a}}^* \}\ d\varepsilon_d^p\ ]$$

with

$$\chi_v = \frac{3\ \underline{\underline{s}}}{4\ p}\ ;\ \chi_d = \frac{\underline{\underline{s}}}{3\ p}\ ; \quad < d\varepsilon_v^p > = d\varepsilon_v^p \text{ if } d\varepsilon_v{}^p < 0 \text{ and zero otherwise}$$

Initialization of the hardening parameter a*

In the context of triaxial variables, for a material deposited by sedimentation and normally consolidated, Wheeler et al propose:

$$\alpha^\circ = \frac{\eta_{Ko}{}^2 + 3\eta_{Ko} - M^2}{3} \quad \text{with} \quad \eta_{Ko} = \frac{3(1 - K_o)}{1 + 2\ K_o}$$

For the scalar parameter β, they propose to take :

$$\beta = \frac{3\ (4\ M^2 - 4\ \eta_{Ko}{}^2 - 3\eta_{Ko})}{8\ (\eta_{Ko}{}^2 + 2\eta_{Ko} - M^2)}$$

No value is recommended for μ.

The three-dimensional formulation has been implemented in CESAR-LCPC. The parameters β and μ being scalar, we can keep the previously defined values.

On the other hand, it is necessary to specify the initial value of the tensor a *.

We admit that the initial stress state is geostatic, defined by:

$$\underline{\underline{\sigma}} = -\sigma_h (\underline{e}_x \otimes \underline{e}_x + \underline{e}_y \otimes \underline{e}_y) - \sigma_v \underline{e}_z \otimes \underline{e}_z = -\sigma_v [\underline{e}_z \otimes \underline{e}_z + K_o (\underline{e}_x \otimes \underline{e}_x + \underline{e}_y \otimes \underline{e}_y)]$$

The deviatoric part of the stress tensor is given by:

$$\underline{\underline{s}} = -2 \sigma_v (1-K_o)/3 [\underline{e}_z \otimes \underline{e}_z - 1/2 (\underline{e}_x \otimes \underline{e}_x + \underline{e}_y \otimes \underline{e}_y)]$$

The deviatoric tensor a* is looked for under the form:

$$\underline{\underline{a}}^* = a [\underline{e}_z \otimes \underline{e}_z - 1/2 (\underline{e}_x \otimes \underline{e}_x + \underline{e}_y \otimes \underline{e}_y)]$$

Assuming that the actual state results from a sedimentary deposit, in which horizontal strain are zero , leads to :

$$d\varepsilon^p_{xx} = 0$$

Which yields, given the flow rule and the derivatives of the criterion:

$$3 s_{xx} - 3 p (-a/2) + (\underline{\underline{a}}^* : \underline{\underline{s}}) + M^2 (p_m - 2p)/3 - 1/2 \underline{\underline{a}}^* : \underline{\underline{a}}^* p_m = 0$$

or:

$$\sigma_v (1-K_o) + 3 p a/2 - a \sigma_v (1-K_o) + M^2 (p_m - 2p)/3 - 1/2 (3/2 a^2) p_m = 0$$

If the stress state is on the yield surface, the following holds

$$f = 0 \Rightarrow p_m = p + \frac{3/2 (\underline{\underline{s}} - p \underline{\underline{a}}^*) : (\underline{\underline{s}} - p \underline{\underline{a}}^*)}{p (M^2 - 3/2 \underline{\underline{a}}^* : \underline{\underline{a}}^*)} = p + \frac{3/2 \underline{\underline{s}} : \underline{\underline{s}} - 3 p \underline{\underline{a}}^* : \underline{\underline{s}} + 3/2 p^2 \underline{\underline{a}}^* : \underline{\underline{a}}^*}{p (M^2 - 3/2 \underline{\underline{a}}^* : \underline{\underline{a}}^*)}$$

$$p_m = p + \frac{[\sigma_v (1-K_o)]^2 + 3 p a \sigma_v (1-K_o) + 9/4 p^2 a^2}{p (M^2 - 9/4 a^2)}$$

Since $p = -1/3 \operatorname{tr} \underline{\underline{\sigma}} = \sigma_v (1+2K_o)/3$, and introducing the stress ratio $\eta = q/p = \frac{\sigma_v (1-K_o)}{\sigma_v (1+2K_o)/3} = \frac{3(1-K_o)}{(1+2K_o)}$, one gets:

$$a = 2(M^2 - \eta^2)/9 - 2\eta/3$$

which provides a practical way of computing the initial value of $\underline{\underline{a}}^*$ for a normally consolidated clay.

The computation of χ_v and χ_d is not possible if p tends towards zero. One can replace the expressions above by:

$$\chi_v = \frac{3 \underline{\underline{s}}}{4 \operatorname{Max}(p, p_o)} \; ; \; \chi_d = \frac{\underline{\underline{s}}}{3 \operatorname{Max}(p, p_o)}$$

15.3. von Mises criterion with Prager non linear kinematic hardening

To study the cyclic behavior of a pile, specific models were implemented to describe a progressive accumulation of plastic deformations. The simplest model in the literature is a nonlinear kinematic hardening law, called model of Prager. The succinct presentation which follows largely repeats the principles set out in Besson et al (2001).

We use a criterion of the type:

$$F(\underline{\underline{\sigma}}, \underline{\underline{X}}) = f(\underline{\underline{\sigma}} - \underline{\underline{X}}) \leq 0$$

where $\underline{\underline{X}} = C \underline{\underline{\varepsilon}}^p$ and f is the von Mises yield function $f(\sigma) = \frac{1}{2} \underline{\underline{s}} : \underline{\underline{s}} - k^2$ where $\underline{\underline{s}}$ is the deviatoric part of $\underline{\underline{\sigma}}$.

The hardening law is given by:

$$\dot{\underline{\underline{X}}} = \frac{2}{3} C \; \dot{\underline{\underline{\varepsilon}}}^p \; - D \; X \; \dot{p}$$

with :

$\dot{p} = (2/3 \; \dot{\underline{\underline{\varepsilon}}}^p : \dot{\underline{\underline{\varepsilon}}}^p)^{1/2}$, where C and D are scalar.

Note : the approach was also tested with a different plasticity criterion, that of Drucker Prager.

Partial derivatives of the yield function

$$\frac{\partial F}{\partial \underline{\underline{\sigma}}} (\underline{\underline{\sigma}}, \underline{\underline{X}}) = \frac{\partial f}{\partial \underline{\underline{\sigma}}} (\underline{\underline{\sigma}} - \underline{\underline{X}})$$

The derivative of F with respect to $\underline{\underline{\sigma}}$ is equal to that of f (von Mises or Drucker Prager).

15.4. Yield functions depending on the third invariant

15.4.1. Invariants and their derivatives

Plastic computations use the derivatives of the criterion with respect to the stress tensor; the difficulty therefore consists here in knowing how to calculate the derivatives of the third invariant of a tensor with respect to this tensor. The invariants are defined as follows:

$I_1 = \text{tr}\ \sigma$; $J_2 = 1/2\ \text{tr}\ (s^2) = 1/2\ s:s$; $J_3 = 1/3\ \text{tr}\ (s^3)$

$\rho = \sqrt{2 J_2}$; $q = \sqrt{3/2\ J_2}$; $\cos(3\theta) = 3\sqrt{3}\ J_3 / (2\ J_2\text{\^{}}(3/2))$

Bigoni and Piccolroaz have provided the expression of their derivatives:

$\partial I_1/\partial\sigma = 1$; $\partial J_2/\partial\sigma = s$; $\partial J_3/\partial\sigma = s^2 - 1/3\ \text{tr}\ s^2\ 1$

$\partial\rho/\partial J_2 = 1/\rho$; $\partial\theta/\partial\sigma = -9\ [s^2 - \text{tr}(s^2)/3\ 1 - q/3\ \cos(3\theta)\ s] / [2\ q^3 \sin(3\ \theta)\ ]$

Theses formula are useful to compute the derivatives of yield functions depending on the third invariant.

15.4.2. Yield function of the “HISS” model

The HISS model (for Hierarchical Incremental Single Surface) is an elastoplastic model for geomaterials, taken from the work of Desai (see for example Shao and Desai (2000)). It has the advantage of being based on only one smooth yield surface, but it takes into account at the same time deviatoric and volumetric plastic strains, and distinguishes shear in compression and in extension.

The expression of the criterion retained here among the different variants is as follows:

$$f(\underline{\underline{\sigma}}) = \frac{J_{2D}}{p_a^2} - \frac{-\alpha\ (J_1^*/p_a)^n + \gamma\ (J_1^*/p_a)^2}{\sqrt{1 - \beta S}}$$

with n>2 and $S = \frac{\sqrt{27}}{2} \frac{J_{3D}}{J_{2D}^{3/2}}$; $J_1^* = J_1 + 3R$

In these expressions, J1 is the first invariant of the stress tensor, and J_{2D} and J_{3D} the second and third invariants of the stress deviator. The formulas defining the invariants are given by Desai (1980):

$$J_1 = \text{tr}\,\underline{\underline{\sigma}}\ ;\ J_2 = \frac{1}{2}\text{tr}\,\underline{\underline{\sigma}}^2 = \frac{1}{2}\sigma_{ij}\sigma_{ji}\ ;\ J_3 = \frac{1}{3}\text{tr}\,\underline{\underline{\sigma}}^3 = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}$$

γ, β, n, R, α_0 and p_a are parameters provided by the user.

Partial derivatives of the yield function :

$$\frac{\partial f}{\partial \underline{\underline{\sigma}}} = \frac{\partial f}{\partial J_1}\frac{\partial J_1}{\partial \underline{\underline{\sigma}}} + \frac{\partial f}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \underline{\underline{\sigma}}} + \frac{\partial f}{\partial J_{3D}}\frac{\partial J_{3D}}{\partial \underline{\underline{\sigma}}}$$

With

$$\frac{\partial J_1}{\partial \underline{\underline{\sigma}}} = -\underline{\underline{1}}\ ;\ \frac{\partial J_{2D}}{\partial \underline{\underline{\sigma}}} = \underline{\underline{s}};\ \frac{\partial J_{3D}}{\partial \underline{\underline{\sigma}}} = -\left(\underline{\underline{s}}^2 - \frac{1}{3}\,\text{tr}\,\underline{\underline{s}}^2\underline{\underline{1}}\right)$$

$$\frac{\partial f}{\partial J_1} = -\frac{-\alpha n\ (J_1^*/p_a)^{n-1} + 2\gamma J_1^*/p_a}{p_a\sqrt{1-\beta S}}$$

$$\frac{\partial f}{\partial J_{2D}} = \frac{1}{p_a^2} - \frac{\beta K}{2(1-\beta S)^{3/2}}\frac{\partial S}{\partial J_{2D}} = \frac{1}{p_a^2} + \frac{3}{2}\frac{\beta K}{2(1-\beta S)^{3/2}}\frac{S}{J_{2D}}$$

$$\frac{\partial f}{\partial J_{3D}} = -\frac{\beta K}{2(1-\beta S)^{3/2}}\frac{\partial S}{\partial J_{3D}} = -\frac{\beta K}{2(1-\beta S)^{3/2}}\frac{\sqrt{27}}{2\,J_{2D}^{3/2}}$$

where

$$K = -\alpha\ (J_1^*/p_a)^n + \gamma\ (J_1^*/p_a)^2$$

The computation of partial derivatives poses problem when the stress state $\underline{\underline{\sigma}}$ is isotropic because one divides by null quantities (J_{2D}). The code has been changed as follows:

If $J_{2D} = 0$, we let $\frac{\partial f}{\partial J_{2D}} = \frac{\partial f}{\partial J_{3D}} = 0$

Similarly $\frac{\partial f}{\partial J_1}$ Depends on $(J_1)^n$ where n is a real number. If J_1 is zero, we let $\frac{\partial f}{\partial J_{1D}}$ =0.

15.5. Note on the numerical treatment of anisotropic elastic models

We are interested in the treatment of orthotropic elasticity. The orthotropy axis is axis 2. The behaviour is written, in the local coordinate system:

$$\begin{bmatrix}\varepsilon_{33}\\ \varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_1/E_1 & -\nu_2/E_2 & 0 & 0 & 0\\ -\nu_1/E_1 & 1/E_1 & -\nu_2/E_2 & 0 & 0 & 0\\ -\nu_2/E_2 & -\nu_2/E_2 & 1/E_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu_1)/E_1 \end{bmatrix} . \begin{bmatrix}\sigma_{33}\\ \sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{bmatrix}$$

Or, inversely :

$$\begin{bmatrix}\sigma_{33}\\ \sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{bmatrix} = \begin{bmatrix} cE_1\dfrac{[1-\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\dfrac{[\nu_1+\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\nu_2 & 0 & 0 & 0 \\ cE_1\dfrac{[\nu_1+\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\dfrac{[1-\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\nu_2 & 0 & 0 & 0 \\ cE_1\nu_2 & cE_1\nu_2 & cE_2\,(1-\nu_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{E_1}{(2(1+\nu_1))} \end{bmatrix} \begin{bmatrix}\varepsilon_{33}\\ \varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix}$$

with $c = 1-\nu_1-2\nu_2^2E_1/ E_2$

15.5.1. Plane strain computations

In plane strain : $\varepsilon_{33}= \varepsilon_{13}= \varepsilon_{23}= 0$ so one only has to write :

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = M_L \,.\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix} \qquad \text{with} \qquad M_L = \begin{bmatrix} cE_1\dfrac{[1-\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\nu_2 & 0 \\ cE_1\nu_2 & cE_2\,(1-\nu_1) & 0 \\ 0 & 0 & G \end{bmatrix}$$

We want to do the calculation in the axes of a Cartesian coordinate system ($\underline{e}_x$, $\underline{e}_y$, $\underline{e}_z$). The orthotropy axis (i.e. axis 2 of the local coordinate system), makes the angle $\alpha + \pi / 2$ with the horizontal:

$$\underline{e}_1 = \cos\alpha\underline{e}_x + \sin\alpha\underline{e}_y \quad ; \qquad \underline{e}_3 = \underline{e}_z \qquad \underline{e}_2 = -\sin\alpha\underline{e}_x + \cos\alpha\underline{e}_y$$

In particular, the following tensorial products are obtained:

$$\underline{e}_1\otimes\underline{e}_1= \cos^2\alpha\underline{e}_x\otimes\underline{e}_x + \sin\alpha\cos\alpha\,(\underline{e}_x\otimes\underline{e}_y + \underline{e}_y\otimes\underline{e}_x) + \sin^2\alpha\, e_y\otimes\underline{e}_y$$

$$\underline{e}_2\otimes\underline{e}_1= -\sin\alpha\cos\alpha\underline{e}_x\otimes\underline{e}_x -\sin^2\alpha\underline{e}_x\otimes\underline{e}_y +\cos^2\alpha\underline{e}_y\otimes\underline{e}_x +\sin\alpha\cos\alpha\underline{e}_y\otimes\underline{e}_y$$

$$\underline{e}_1\otimes\underline{e}_2= -\sin\alpha\cos\alpha\underline{e}_x\otimes\underline{e}_x +\cos^2\alpha\underline{e}_x\otimes\underline{e}_y -\sin^2\alpha\underline{e}_y\otimes\underline{e}_x +\sin\alpha\cos\alpha\underline{e}_y\otimes\underline{e}_y$$

$$\underline{e}_2\otimes\underline{e}_2= \sin^2\alpha\underline{e}_x\otimes\underline{e}_x - \sin\alpha\cos\alpha\,(\underline{e}_x\otimes\underline{e}_y + \underline{e}_y\otimes\underline{e}_x) + \cos^2\alpha\underline{e}_y\otimes\underline{e}_y$$

One can thus express the components of the tensor of strain or the tensor of stresses in the axes of the total reference ($\underline{e}_x,\underline{e}_y,\underline{e}_z$) according to their components in the local reference ($\underline{e}_1,\underline{e}_2,\underline{e}_3$)

$$\underline{\underline{\varepsilon}} = \varepsilon_{11}\, \underline{e}_1\otimes\underline{e}_1 + \varepsilon_{22}\underline{e}_2\otimes\underline{e}_2 + \varepsilon_{12}\,(\underline{e}_1\otimes\underline{e}_2 + \underline{e}_2\otimes\underline{e}_1)$$

$$\underline{\underline{\sigma}} = \sigma_{11}\underline{e}_1\otimes\underline{e}_1 + \sigma_{22}\underline{e}_2\otimes\underline{e}_2 + \sigma_{12}\,(\underline{e}_1\otimes\underline{e}_2 + \underline{e}_2\otimes\underline{e}_1)$$

One finds

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix} = T(\alpha)\;.\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix} \qquad \text{et} \qquad \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = P(\alpha)\;.\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} =$$

with :

$$T(\alpha) = \begin{bmatrix} \cos2\alpha & \sin2\alpha & -\sin\alpha\cos\alpha \\ \sin2\alpha & \cos2\alpha & \sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & \cos2\alpha-\sin2\alpha \end{bmatrix}$$

and

$$P(\alpha) = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix}$$

For the calculation of the stiffness matrix, the energy is given, in matrix terms, by

$$W = \frac{1}{2}\,\sigma_L \varepsilon_L$$

where σ_L and ε_L are the vectors of the components of the stresses and strains in the local base :

$$\sigma_L = M_L \,.\, \varepsilon_L$$

$$W = \frac{1}{2}\,\sigma_L .\, \varepsilon_L = \frac{1}{2}\,{}^t\varepsilon_L \,.\, M_L .\, \varepsilon_L$$

To get back to the global axes, we introduce the matrix T and we find:

$$\varepsilon_L = T(-\alpha)\,.\,\varepsilon_G$$

$$W = \frac{1}{2}\,{}^t\varepsilon_G \,.\, {}^tT(-\alpha).M_L T(-\alpha).\varepsilon_G$$

For the stresses, the following holds :

$$\sigma_G = P(\alpha)\,.\,\sigma_L = P(\alpha).\,M_L \,.\, \varepsilon_L = P(\alpha).\,M_L \,.\, T(-\alpha).\varepsilon_G$$

NB : it is readily seen that ${}^tT(-\alpha)=P(\alpha)$.

$$\sigma_{zz}=\sigma_{33} = cE_1\frac{[\nu_1+\nu_2^2E_1/E_2]}{[1+\nu_1]}\varepsilon_{11} + cE_1\nu_2\;\varepsilon_{22}$$

where

$$\varepsilon_{11} = \cos^2\alpha\;\;\varepsilon_{xx} + \sin^2\alpha\;\;\varepsilon_{yy} + \sin\alpha\,\cos\alpha\;2\,\varepsilon_{xy}$$

$$\varepsilon_{22} = \sin^2\alpha\;\;\varepsilon_{xx} + \cos^2\alpha\;\;\varepsilon_{yy} - \sin\alpha\,\cos\alpha\;2\,\varepsilon_{xy}$$

Note : as regards the computations of plastic strain, the numerical treatment is entirely carried out in 3D.

15.5.2. Three dimensional condition

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_2/E_2 & -\nu_1/E_1 & 0 & 0 & 0\\ -\nu_2/E_2 & 1/E_2 & -\nu_2/E_2 & 0 & 0 & 0\\ -\nu_1/E_1 & -\nu_2/E_2 & 1/E_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu_1)/E_1 \end{bmatrix} . \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{bmatrix}$$

$$.\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{bmatrix} = \begin{bmatrix} cE_1\frac{[1-\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\nu_2 & cE_1\frac{[\nu_1+\nu_2^2E_1/E_2]}{[1+\nu_1]} & 0 & 0 & 0\\ cE_1\nu_2 & cE_2\,(1-\nu_1) & cE_1\nu_2 & 0 & 0 & 0\\ cE_1\frac{[\nu_1+\nu_2^2E_1/E_2]}{[1+\nu_1]} & cE_1\nu_2 & cE_1\frac{[1-\nu_2^2E_1/E_2]}{[1+\nu_1]} & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{E_1}{2[1+\nu_1]} \end{bmatrix} . \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix}$$

Again one uses the following relations

$$\underline{e}_1 \otimes \underline{e}_3 = \cos\alpha\,\underline{e}_x \otimes \underline{e}_z + \sin\alpha\,\underline{e}_y \otimes \underline{e}_z$$

$\underline{e}_2 \otimes \underline{e}_3 = -\sin\alpha\, \underline{e}_x \otimes \underline{e}_z + \cos\alpha\, \underline{e}_y \otimes \underline{e}_z$

$\underline{e}_3 \otimes \underline{e}_3 = \underline{e}_z \otimes \underline{e}_z$

Which provides

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix} = T_3(\alpha). \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} \quad \text{et} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = P_3(\alpha).. \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix}$$

where

$$T_3(\alpha) = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 0 & -\sin\alpha\cos\alpha & 0 & 0 \\ \sin^2\alpha & \cos^2\alpha & 0 & \sin\alpha\cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & 0 & \cos^2\alpha-\sin^2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$P_3(\alpha) = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 0 & -2\sin\alpha\cos\alpha & 0 & 0 \\ \sin^2\alpha & \cos^2\alpha & 0 & 2\sin\alpha\cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & -\sin\alpha\cos\alpha & 0 & \cos^2\alpha-\sin^2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

15.6. Note on the formulation of hardening laws for geomaterials

Many constitutive models have been constructed from triaxial test results, in a very specific context where the principal directions of the stress and strain tensors are identical, and fixed in space. The extension to the three-dimensional case of the formulation established in a particular framework poses various difficulties. One of them is the choice of the variables to be used to take into account a contribution of the deviatoric deformations in the hardening law.

15.6.1. Formulation in triaxial variables

Many models assume that the plasticity criterion is a function of the mean stress p and the deviator q, defined by: $p = -\frac{1}{3}\,(\mathrm{tr}\,\underline{\underline{\sigma}})$, $q = (\frac{3}{2}\,\underline{\underline{s}} : \underline{\underline{s}})^{1/2}$

During a triaxial revolution test, two of the principal stresses are equal (the two horizontal stresses), and the stresses and strains (plastic or elastic) have fixed directions. If we note 1 the vertical direction, the tensors therefore have the following form:

$$\underline{\underline{\sigma}} = \sigma_1\, \underline{e}_1 \otimes \underline{e}_1 + \sigma_3\, (\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3)$$

$$\underline{\underline{\varepsilon}} = \varepsilon_1 \underline{e}_1 \otimes \underline{e}_1 + \varepsilon_3\, (\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3)$$

In the case of triaxial compression : $\sigma_1 < \sigma_3 < 0$, so that :

$$p = -(\sigma_1 + 2\,\sigma_3)/3$$

$$q = -(\sigma_1 - \sigma_3)$$

It is readily seen that the intrinsic dissipation $\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p$ is equal to :

$$\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p = \sigma_1 \dot{\varepsilon}_1^p + 2\,\sigma_3 \dot{\varepsilon}_3^p = p\,\dot{\varepsilon}_v^p + q\,\dot{\varepsilon}_d^p$$

where ε_v^p and ε_d^p denote the volumetric and deviatoric part of the plastic strain tensor:

$$\varepsilon_v^p = -(\varepsilon_1 + 2\,\varepsilon_3)$$

$$\varepsilon_d^p = -\frac{2}{3}\,(\varepsilon_1 - \varepsilon_3) \qquad (1)$$

The plasticity criterion being denoted by f(p,q), the expression obtained above for the intrinsic dissipation leads to describe (in the associated case) the plastic flow regime by a law such as (voir par exemple Muir Wood, 1990) :

$$\dot{\varepsilon}_v^p = \dot{\lambda}\frac{\partial f}{\partial p}$$

$$\dot{\varepsilon}_d^p = \dot{\lambda}\frac{\partial f}{\partial q}$$

where $\dot{\lambda}$ is the plastic multiplier, whose value is determined by the consistency condition $\dot{f}=0$.

In the case of the modified Cam Clay model for instance, the yield function is given by:

$$f\,(p,q) = q^2 + M^2\, p\,(p_c - p) \qquad (2)$$

and the hardening parameter p_c depends on the plastic void ratio. Its evolution is often described by an exponentail relation such as:

$$p_c = p_c{}^{\circ}\ \exp\left[\frac{1+e_o}{\lambda-\kappa}\,\varepsilon_v^p\right]$$

where e_o is the initial void ratio and λ and κ the slopes of the e – ln p curves (in initial consolidation and in unloading reloading) during an isotropic compression test. From a numerical point of view, it is necessary to compute the hardening modulus H, defined by $H\dot{\lambda} = \frac{\partial f}{\partial \underline{\underline{\sigma}}}:\dot{\underline{\underline{\sigma}}}$. Using the consistency condition, the flow rule and the hardening rule, one gets:

$$H = -\frac{\partial f}{\partial p_c}\dot{p}_c / \dot{\lambda} = -\frac{\partial f}{\partial p_c}\frac{\partial p_c}{\partial \varepsilon_v^p}\frac{\partial f}{\partial p}$$

For a non associated flow rule, it would be advantageous to use the following more general formulation:

$$H = -\frac{\partial f}{\partial p_c}\dot{p}_c / \dot{\lambda} = -\frac{\partial f}{\partial p_c}\frac{\partial p_c}{\partial \varepsilon_v^p}\,\mathrm{tr}\left(\frac{\partial g}{\partial \underline{\underline{\sigma}}}\right)$$

15.6.2. Introduction the deviatoric strain in the hardening law

The modified Cam-Clay model is used to model the shear failure of a soil sample during an undrained test with a triaxial device. During a shear in undrained condition, the point (p', q) moves along the yield surface until it reaches the critical state; the sample then deforms at constant volume under a constant stress state. One cannot apply a shear stress higher than a maximum value q_{max} which one calls "undrained shear strength".

For the current values of the parameters of the model, the size of the elastic domain practically does not vary during the undrained shear, and the undrained shear strength is calculated simply according to the initial size p_c of the domain (i.e. of the preconsolidation pressure): we find $q_{max} \approx M\, p_c /2$.

For certain materials, in particular loose sands, one observes that the stress path is more complex: after a first phase during which the average stress decreases (which one can account for with a surface and a law of hardening of the type of those of the Cam-Clay model), an increase in the average stress is observed: the loading point goes up along a generally almost linear rupture curve.

The Nova (1982) model is one of the models developed to reproduce this behavior. The load surface is different from that of the Cam-Clay model, but can undergo isotropic hardening and its size depends on a parameter analogous to the pre-consolidation pressure p_c of the Cam-Clay model. One does not discuss here the interest of this surface of load, but the contribution of the law of hardening proposed by Nova. This law makes the size of the load surface depend not only on the volume part of the plastic deformation, but also on the deviatoric plastic deformation. So we have a relation of the type:

$$p_c = p_c(\varepsilon_v^p, \varepsilon_d^p) = p_c^{\circ}\ \exp[\alpha\varepsilon_v^p + \beta\varepsilon_v^p]$$

The introduction of this type of hardening law in the modified Cam-Clay model makes it possible to obtain, qualitatively, an undrained stress path which presents a change of direction: the mean stress decreases at first then increases. For the practical implementation of this law in a calculation code, it is convenient to establish the expression of the hardening module H:

$$H\dot{\lambda} = \frac{\partial f}{\partial \underline{\underline{\sigma}}}:\dot{\underline{\underline{\sigma}}} = -\frac{\partial f}{\partial p_c}\dot{p}_c = -\frac{\partial f}{\partial p_c}\left\{\frac{\partial p_c}{\partial \varepsilon_v^p}\dot{\varepsilon}_v^p + \frac{\partial p_c}{\partial \varepsilon_d^p}\dot{\varepsilon}_d^p\right\} \rightarrow H = -\frac{\partial f}{\partial p_c}\left\{\frac{\partial p_c}{\partial \varepsilon_v^p}\frac{\partial f}{\partial p} + \frac{\partial p_c}{\partial \varepsilon_d^p}\frac{\partial f}{\partial q}\right\}$$

For the modified Cam-Clay model, one gets:

$$H = M^2 p\ \left\{ M^2 (2p-p_c)\frac{\partial p_c}{\partial \varepsilon_v^p} + 2q\frac{\partial p_c}{\partial \varepsilon_d^p}\right\} \qquad (3)$$

15.6.3. Extension to three dimensional conditions

The previous developments take place in the context of the triaxial test:

- two of the principal stresses are equal;

- the principal directions of stresses and deformations do not change.

To apply the model to the study of a real structure (even simple), it is necessary to propose a general formulation, usable when the principal stresses and the principal directions of the stress tensor are arbitrary. We keep the expressions of p and q given above. It is reasonable to adopt the following definition for ε_v^p:

$$\varepsilon_v^p = -\,\mathrm{tr}\,\underline{\underline{\varepsilon}}^p$$

It must be noted that it is no longer possible to put the intrinsic dissipation in such a simple form as before:

$$\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p \neq p\,\dot{\varepsilon}_v^p + q\,\dot{\varepsilon}_d^p$$

If the plastic potential depends on p and q, one gets:

$$\dot{\underline{\underline{\varepsilon}}}^p = \dot{\lambda}\frac{\partial g}{\partial \underline{\underline{\sigma}}} = \dot{\lambda}\,[\frac{\partial g}{\partial p}\frac{\partial p}{\partial \underline{\underline{\sigma}}} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial \underline{\underline{\sigma}}}] = \dot{\lambda}\,[\,-\,1/3\,\frac{\partial g}{\partial p}\,\underline{\underline{1}} + \frac{\partial g}{\partial q}\,3\underline{\underline{s}}/2q]$$

It remains to formulate the hardening law and the calculation of the hardening modulus. We denote by $\underline{\underline{\varepsilon}}_d$ the deviatoric plastic strain, defined by:

$$\varepsilon_d^p = (\frac{2}{3}\,\underline{\underline{\varepsilon}}_d : \underline{\underline{\varepsilon}}_d)^{1/2}$$

This definition makes it possible to find the desired value in the context of the triaxial test. For the calculation of the hardening module, we obtain:

$$H\,\dot{\lambda} = \frac{\partial f}{\partial \underline{\underline{\sigma}}}:\dot{\underline{\underline{\sigma}}} = -\,\frac{\partial f}{\partial p_c}\,\dot{p}_c = -\,\frac{\partial f}{\partial p_c}\,\{\frac{\partial p_c}{\partial \varepsilon_v^p}\dot{\varepsilon}_v^p + \frac{\partial p_c}{\partial \varepsilon_d^p}\dot{\varepsilon}_d^p\}$$

It is readily seen that:

$$\dot{\varepsilon}_v^p = -\,\mathrm{tr}\,\dot{\underline{\underline{\varepsilon}}}^p = -\,\dot{\lambda}\,\mathrm{tr}\,\frac{\partial g}{\partial \underline{\underline{\sigma}}} = \dot{\lambda}\frac{\partial g}{\partial p}$$

$$2\,\varepsilon_d^p\dot{\varepsilon}_d^p = \frac{4}{3}\,\underline{\underline{\varepsilon}}_d^p : \dot{\underline{\underline{\varepsilon}}}_d^p$$

$$\dot{\underline{\underline{\varepsilon}}}_d^p = \dot{\lambda}\,\frac{\partial g}{\partial q}\frac{\partial q}{\partial \underline{\underline{\sigma}}}$$

Which leads to:

$$H = -\,\frac{\partial f}{\partial p_c}\,\{\frac{\partial p_c \partial g}{\partial \varepsilon_v^p \partial p} + \frac{2\partial p_c \partial g}{3\partial \varepsilon_d^p \partial q}\underline{\underline{\varepsilon}}_d^p : \frac{\partial q}{\partial \underline{\underline{\sigma}}}\;/\;\varepsilon_d^p\,\}$$

Given that $2\,q\,\frac{\partial q}{\partial \underline{\underline{\sigma}}} = 3\,\underline{\underline{s}}$

$$H = -\,\frac{\partial f}{\partial p_c}\,\{\frac{\partial p_c \partial g}{\partial \varepsilon_v^p \partial p} + \frac{\partial p_c \partial g\,\underline{\underline{\varepsilon}}_d^p : \underline{\underline{s}}}{\partial \varepsilon_d^p \partial q\;\;q\;\varepsilon_d^p}\,)\,\}$$

For the modified Cam Clay model and an associated flow rule, on finds:

$$H = M^2\,p\,\{\frac{\partial p_c}{\partial \varepsilon_v^p}\,M^2\,(2p-p_c) + 2\,\frac{\partial p_c\,\underline{\underline{\varepsilon}}_d^p : \underline{\underline{s}}}{\partial \varepsilon_d^p\;\;\varepsilon_d^p}\,\} \qquad (4)$$

If $\varepsilon_d^p = 0$, this expression reduces to:

$$H = M^2 p \{ \frac{\partial p_c}{\partial \varepsilon_v^p} M^2 (2p-p_c) + \frac{\partial p_c}{\partial \varepsilon_d^p} \sqrt{\frac{2}{3} \left[\frac{\partial f}{\partial \underline{\underline{\sigma}}}\right]_d : \left[\frac{\partial f}{\partial \underline{\underline{\sigma}}}\right]_d} \}$$

For a triaxial test, the last formula is identical to the one provided in 15.6.2. But, if a given plastic strain has been generated on a given stress path, and the material has been unloaded, then for subsequent loading, both formula give different values of the hardening modulus.

More precisely, when the directions of the principal stresses rotate, with the second formula, a soil element which has undergone plastic deformations and which has been unloaded gives different responses according to the directions in which it is stressed, which constitutes a (summary) form of induced anisotropy (there is information stored in the main directions of $\underline{\underline{\varepsilon}}_d^p$). The yield surface remains isotropic, and the plastic strain increments remain coaxial with the stress state. On the other hand, of the hardening modulus, and therefore the apparent (tangent) stiffness of the soil when reloading in plastic regime, are different. The choice to include among the variables of the hardening law the plastic deviatoric deformation defined by $\varepsilon_d^p = (\frac{2}{3} \underline{\underline{\varepsilon}}_d : \underline{\underline{\varepsilon}}_d)^{1/2}$ therefore gives the model undesirable properties. The initial formulation is better, insofar as it does not show poorly controlled anisotropy. It will be preferred, although it has the following drawback. If the scalar quantity ε_d^p is defined by:

$$\dot{\varepsilon}_d^p = \dot{\lambda} \frac{\partial f}{\partial q}$$

it is not simply related with the plastic strain tensor (or its variations).

15.6.4. Other formulations for the deviatoric strain variable

Some authors (notably Shao and Desai (2000)) propose other variables instead of ε_v^p and ε_d^p to formulate the hardening law:

$$p_c = p_c(\xi_v, \xi_d)$$

where the evolution of variables ξ_v and ξ_d depend on the variations of the plastic strain tensor.

For instance, they give :

$$\dot{\xi}_d^p = (\frac{2}{3} \dot{\underline{\underline{\varepsilon}}}_d^p : \dot{\underline{\underline{\varepsilon}}}_d^p)^{1/2}$$

Using the flow rule, one gets:

$$\dot{\underline{\underline{\xi}}}_d^p = \dot{\lambda} \left[\frac{\partial g}{\partial \underline{\underline{\sigma}}} - \frac{1}{3} \mathrm{tr} \frac{\partial g}{\partial \underline{\underline{\sigma}}} \underline{\underline{1}} \right]$$

$$\dot{\xi}_d^p = \dot{\lambda} \sqrt{\frac{2}{3} \left[\frac{\partial g}{\partial \underline{\underline{\sigma}}}\right]_d : \left[\frac{\partial g}{\partial \underline{\underline{\sigma}}}\right]_d} = \dot{\lambda} \sqrt{\frac{2}{3} \{ \frac{\partial g}{\partial \underline{\underline{\sigma}}} : \frac{\partial g}{\partial \underline{\underline{\sigma}}} - \frac{1}{3} (\mathrm{tr} \frac{\partial g}{\partial \underline{\underline{\sigma}}})^2 \}} \qquad (5)$$

This formulation does not suffer from the same weakness as the previous one: the variation of $\dot{\xi}_d^p$ does not depend on the orientation of the principal directions of $\underline{\underline{\sigma}}$. For the volumetric part of the plastic strain, Shao and Desai (2000) propose a formulation with which a plastic increase in volume does not induce a hardening of the yield surface:

$$\dot{\xi}_v^p = \mathrm{Max} (0, - \frac{1}{3} \mathrm{tr} \, \dot{\underline{\underline{\varepsilon}}}^p) = \dot{\lambda} \mathrm{Max} (0, - \frac{1}{3} \mathrm{tr} \frac{\partial g}{\partial \underline{\underline{\sigma}}})$$

Eventually the hardening modulus is given by:

$$H = - \frac{\partial f}{\partial p_c} \{ \frac{\partial p_c}{\partial \xi_v^p} \mathrm{Max} (0, - \frac{1}{3} \mathrm{tr} \frac{\partial g}{\partial \underline{\underline{\sigma}}}) + \frac{\partial p_c}{\partial \xi_d^p} \sqrt{\frac{2}{3} \{ \frac{\partial g}{\partial \underline{\underline{\sigma}}} : \frac{\partial g}{\partial \underline{\underline{\sigma}}} - \frac{1}{3} (\mathrm{tr} \frac{\partial g}{\partial \underline{\underline{\sigma}}})^2 \}} \}$$

15.6.5. Conclusion

It is not very common to use the yield surface of the modified Cam-Clay model for loose sands, even by adapting the hardening law, because the "friction" part of the criterion (q / p> M), does not represent their behavior well. The above considerations remain valid for models using a different yield surface, with a more representative friction part (such as the models of Nova (1982), Desai et al (1980), Bigoni and Piccolroaz or Menetrey and Willam): in shcu situations, to take into account a contribution from the deviatoric part of plastic strain, it is highly preferable to use the variable proposed by Shao and Desai (2000).

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